이학 박사 학위논문

The volume and Chern-Simons invariant of a Dehn-filled manifold

(덴-채움 된 다양체의 부피와 천-사이먼즈 불변량)

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지도교수 박 종 일

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윤 석 범

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위 원 장	김 상 현	(인)
부위원장	박 종 일	(인)
위 원	국 웅	(인)
위 원	김 혁	(인)
위 원	박 진 성	(인)

The volume and Chern-Simons invariant of a Dehn-filled manifold

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Seokbeom Yoon

Dissertation Director : Professor Jongil Park

Department of Mathematical Sciences
Seoul National University

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Abstract

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Seokbeom Yoon
Department of Mathematical Sciences
Seoul National University

Based on the work of Neumann, Zickert gave a simplicial formula for computing the volume and Chern-Simons invariant of a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$-representation of a compact 3-manifold with non-empty boundary. Main aim of this thesis is to introduce a notion of deformed Ptolemy assignments (or varieties) and generalize the formula of Zickert to a representation of a Dehn-filled manifold. We also generalize the potential function of Cho and Murakami by applying our formula to an octahedral decomposition of a link complement in the 3-sphere. Also, motivated from the work of Hikami and Inoue, we clarify the relation between Ptolemy assignments and cluster variables when a link is given in a braid position. The last work is a joint work with Jinseok Cho and Christian Zickert.

Key words: Hyperbolic manifold, volume, Chern-Simons invariant, Ptolemy variety, cluster variable.
Student Number: 2013-20236

Contents

Chapter 1

Introduction

For an oriented complete hyperbolic 3-manifold N of finite volume, the *complex volume* of N is given by

$$\mathrm{Vol}_{\mathbb{C}}(N) := \mathrm{Vol}(N) + i\mathrm{CS}(N) \in \mathbb{C}/i\pi^2\mathbb{Z}$$

where Vol and CS denote the volume and Chern-Simons invariant, respectively. See, for instance, [Dup87, NZ85]. More generally, for a boundary parabolic $\mathrm{PSL}(2,\mathbb{C})$-representation ρ of a compact 3-manifold one can define *the complex volume* $\mathrm{Vol}_{\mathbb{C}}(\rho)$ by using the Cheeger-Chern-Simons form defined on the flat $\mathrm{PSL}(2,\mathbb{C})$-bundle. We refer to [GTZ15] for details.

1.1 Deformed Ptolemy assignments

Let N be an oriented compact 3-manifold with non-empty boundary. We fix an ideal triangulation of the interior of N with ideal tetrahedra, say $\Delta_1, \cdots, \Delta_n$. Recall that an ideal tetrahedron Δ with mutually distinct vertices, say $z_0, z_1, z_2, z_3 \in \partial\overline{\mathbb{H}^3}$, is determined up to isometry by the *cross-ratio* (or the shape parameter

parameter)

$$z = [z_0 : z_1 : z_2 : z_3] := \frac{(z_0 - z_3)(z_1 - z_2)}{(z_0 - z_2)(z_1 - z_3)} \in \mathbb{C}\backslash\{0, 1\}$$

where the cross-ratio at each edge of Δ is given by one of $z, z' := \frac{1}{1-z}$, and $z'' := 1 - \frac{1}{z}$ (see Figure 3.1). Due to Thurston [Thu78], it is well-known that whenever the shape parameters satisfy the gluing equations and completeness condition, we obtain a boundary parabolic representation $\rho : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ as a holonomy representation.

The cross-ratios are good parameters for computing the volume but not enough for the complex volume. See, for instance, [Dup87]. However, Neumann [Neu04] showed that computing the complex volume can be achieved by considering two additional integers for each ideal tetrahedron which play a role to adjust branches of logarithm functions as follows.

Definition 1.1.1 ([Neu04])**.** A *flattening* of an ideal tetrahedron with the shape parameter $z \in \mathbb{C}\backslash\{0, 1\}$ is a triple $\alpha = (\alpha^0, \alpha^1, \alpha^2) \in \mathbb{C}^3$ of the form

$$\begin{cases} \alpha^0 &= \log z + p\pi i \\ \alpha^1 &= -\log(1 - z) + q\pi i \\ \alpha^2 &= -\log z + \log(1 - z) - (p + q)\pi i \end{cases}$$

for some $p, q \in \mathbb{Z}$. Alternatively, a *flattening* is a triple $\alpha = (\alpha^0, \alpha^1, \alpha^2) \in \mathbb{C}^3$ satisfying $\alpha^0 + \alpha^1 + \alpha^2 = 0$ and

$$\alpha^0 \equiv \log z, \ \alpha^1 \equiv \log z', \ \alpha^2 \equiv \log z''$$

in modulo πi. Here and throughout the paper, we fix a branch of the logarithm; for actual computation we will use the principal branch having the imaginary part in the interval $(-\pi, \pi]$.

Theorem 1.1.1 ([Neu04])**.** Suppose the interior of a compact 3-manifold N

decomposes into n ideal tetrahedra $\Delta_1, \cdots, \Delta_n$. Then for any collection of flattenings α_j of Δ_j satisfying (i) *parity condition*; (ii) *edge conditon*; (iii) *cusp condition*, we have

$$i\mathrm{Vol}_{\mathbb{C}}(\rho) \equiv \sum_{j=1}^{n} R(\alpha_j) \mod \pi^2\mathbb{Z}$$

where $\rho : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ is a boundary parabolic representation induced from the flattenings and R is the extend Rogers dilogarithm given by

$$R(z; p, q) = \mathrm{Li}_2(z) + \frac{\pi i}{2}(p \log(1-z) + q \log z) + \frac{1}{2}\log(1-z)\log z - \frac{\pi^2}{2}.$$

For simplicity, we here assume that every ideal tetrahedron is positively oriented (see Chapter 3).

Roughly speaking, the edge and cusp conditions are additive versions of the gluing equations and completeness condition (obtained by taking logarithm) in [Thu78], respectively. It follows that if the flattenings satisfy the edge and cusp conditions, then the shape parameters automatically satisfy the gluing equations and completeness condition. We therefore obtain an *induced* boundary parabolic representation $\rho : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ as a holonomy representation. We refer to Chapter 3 for details.

Fattenings satisfying the conditions in Theorem 1.1.1 give us the complex volume but finding such one may be difficult in general. Fortunately, Zickert [Zic09] (see also [GTZ15]) remarkably overcame this potential difficulty through the notion of a Ptolemy assignment (or variety). We here briefly recall his key idea.

Let $\mathcal{T}$ be an ideal triangulation of the interior of N. We denote by $\mathcal{T}^1$ the set of the oriented edges. For an oriented edge $e \in \mathcal{T}^1$ we denote by $-e$ the same edge e with its opposite orientation.

Definition 1.1.2 ([GTZ15])**.** A *Ptolemy assignment* is a set map $c : \mathcal{T}^1 \to$

$\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ satisfying $-c(e) = c(-e)$ for all $e \in \mathcal{T}^1$ and

$$c(l_3)c(l_6) = c(l_1)c(l_4) + c(l_2)c(l_5)$$

for each tetrahedron Δ_j of $\mathcal{T}$, where l_i's are the edges of Δ_j as in Figure 1.1.

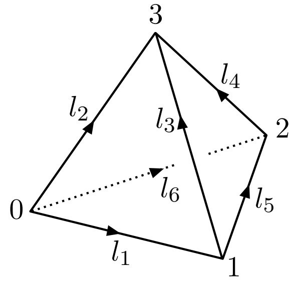


Figure 1.1: An ideal tetrahedron Δ_j of $\mathcal{T}$

A Ptolemy assignment c is *associated* with a boundary parabolic representation $\rho_c : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ up to conjugation. See [GTZ15] or Section 3.2. It also determines the shape parameter of each Δ_j (see [Zic09, Lemma 3.15] or Proposition 3.2.7):

$$z_j = \pm\frac{c(l_1)c(l_4)}{c(l_2)c(l_5)},\ z'_j = \pm\frac{c(l_2)c(l_5)}{c(l_3)c(l_6)},\ z''_j = \pm\frac{c(l_3)c(l_6)}{c(l_1)c(l_4)} \tag{1.1.1}$$

for Figure 1.1 where z_j, z'_j, and z''_j are the cross-ratios at l_3, l_4, and l_2, respectively.

A key idea of [Zic09] is that taking a "logarithm" of the equation (1.1.1) gives us a nice flattening in the sense of Theorem 1.1.1. Namely, if we take a flattening $\alpha_j = (\alpha_j^0, \alpha_j^1, \alpha_j^2)$ of Δ_j as

$$\begin{cases} \alpha_j^0 &= \log c(l_1) + \log c(l_4) - \log c(l_2) - \log c(l_5) \\ \alpha_j^1 &= \log c(l_2) + \log c(l_5) - \log c(l_3) - \log c(l_6) \\ \alpha_j^2 &= \log c(l_3) + \log c(l_6) - \log c(l_1) - \log c(l_4) \end{cases}$$

then these flattenings automatically satisfy the edge and cusp conditions in Theorem 1.1.1. Note that α_j is a flattening, i.e., $\alpha_j^0 + \alpha_j^1 + \alpha_j^2 = 0$ and $\alpha_j^0 \equiv \log z_j$, $\alpha_j^1 \equiv \log z_j'$, $\alpha_j^2 \equiv \log z_j''$ in modulo πi. Moreover, even though the parity condition may fail, it is proved that these flattenings still give the complex volume of ρ_c. Namely,

$$i\mathrm{Vol}_{\mathbb{C}}(\rho_c) \equiv \sum R(\alpha_j) \mod \pi^2\mathbb{Z}.$$

We refer to [Zic09, GTZ15] for details.

1.1.1 Overview

In Chapter 3, we extend the formula of Zickert to a representation that is not necessarily boundary parabolic. We here give an overview. We assume that each boundary component Σ_j of a compact 3-manifold N is a torus with a fixed meridian μ_j and a longitude λ_j for $1 \leqslant j \leqslant h$ where h is the number of the components of ∂N.

In Section 3.2, we suggest a notion of a deformed Ptolemy assignment as a generalization of a Ptolemy assignment. A deformed Ptolemy assignment c determines a representation $\rho_c : \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$ up to conjugation which is not necessarily boundary parabolic. We stress that this is defined in a quite different way from an enhanced Ptolemy assignment in [Zic16].

For $\kappa = (r_1, s_1, \cdots, r_h, s_h)$ we denote by N_κ the manifold obtained from N by Dehn-filling that kills the curve $r_j\mu_j + s_j\lambda_j$ on each boundary torus Σ_j, where (r_j, s_j) is either a pair of coprime integers or the symbol ∞ meaning that we do not fill Σ_j.

Suppose the representation $\rho_c : \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$ factors through $\pi_1(N_\kappa)$ for some κ as a $\mathrm{PSL}(2, \mathbb{C})$-representation. If the manifold N_κ has a boundary, i.e. $(r_j, s_j) = \infty$ for some j, then we further assume that the induced representation $\rho_c : \pi_1(N_\kappa) \to \mathrm{PSL}(2, \mathbb{C})$ is boundary parabolic so that the complex volume of

ρ_c is well-defined. In Section 3.3, we show that the idea of Zickert can be applied to this deformed case, not directly however, so the complex volume of ρ_c can be computed in a similar way (see Theorem 3.3.1). As examples, we compute the complex volumes of several Dehn-filled manifolds obtained from the figure-eight knot complement.

1.2 Potential functions

Let L be a link in S^3 with a fixed diagram and let $N = S^3 \backslash L$. Motivated by the work of Yokota [Yok02], Cho and Murakami [CM13] defined the *potential function* $W(w_1, \cdots, w_n)$ satisfying the following properties: (i) a non-degenerate point $\mathbf{w} = (w_1, \cdots, w_n) \in (\mathbb{C}^\times)^n = (\mathbb{C}\backslash\{0\})^n$ satisfying

$$\exp\left(w_j \frac{\partial W}{\partial w_j}\right) = 1 \quad \text{for all } 1 \leqslant j \leqslant n \tag{1.2.2}$$

corresponds to a boundary parabolic representation $\rho_{\mathbf{w}} : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ (we shall clarify the meaning of a non-degenerate point in Section 4.1); (ii) the complex volume of $\rho_{\mathbf{w}}$ is

$$i\mathrm{Vol}_{\mathbb{C}}(\rho_{\mathbf{w}}) \equiv W_0(\mathbf{w}) \mod \pi^2\mathbb{Z}$$

where the function $W_0(w_1, \cdots, w_n)$ is given by

$$W_0 := W(w_1, \cdots, w_n) - \sum_{j=1}^{n} \left(w_j \frac{\partial W}{\partial w_j}\right) \log w_j.$$

Furthermore, Cho [Cho16a] proved that (iii) any boundary representation $\rho : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ which does not send a meridian of each component of L to the identity matrix is detected by W. Namely, there exists a non-degenerate point $\mathbf{w} \in (\mathbb{C}^\times)^n$ satisfying the equation (1.2.2) such that the corresponding representation $\rho_{\mathbf{w}}$ agrees with ρ up to conjugation.

1.2.1 Overview

In Chapter 4 we extend the potential function to a representation that is not necessarily boundary parabolic. Precisely, we define a *generalized potential function*

$$\mathbb{W}(\mathbf{w},\mathbf{m}) = \mathbb{W}(w_1,\cdots,w_n,m_1,\cdots,m_h),$$

where h is the number of the components of L, and show that it satisfies analogous properties, Theorems 1.2.1, 1.2.2 and 1.2.3, to the potential function W.

We enumerate the components of L by $1 \leqslant i \leqslant h$ and let μ_i and λ_i be a meridian and the canonical longitude of each component, respectively.

Theorem 1.2.1. A non-degenerate point $(\mathbf{w},\mathbf{m}) \in (\mathbb{C}^\times)^{n+h}$ satisfying

$$\exp\left(w_j \frac{\partial \mathbb{W}}{\partial w_j}\right) = 1 \quad \text{for all } 1 \leqslant j \leqslant n \tag{1.2.3}$$

corresponds to a representation $\rho_{\mathbf{w},\mathbf{m}} : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ up to conjugation such that the eigenvalues of $\rho_{\mathbf{w},\mathbf{m}}(\mu_i)$ are m_i and m_i^{-1} up to sign for all $1 \leqslant i \leqslant h$.

Theorem 1.2.2. Let $\rho : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation such that $\rho(\mu_i) \neq \pm I$ for all $1 \leqslant i \leqslant h$. If ρ admits a $\mathrm{SL}(2,\mathbb{C})$-lifting, then there exists a non-degenerate point $(\mathbf{w},\mathbf{m})$ satisfying the equation (1.2.3) such that the corresponding representation $\rho_{\mathbf{w},\mathbf{m}}$ agrees with ρ up to conjugation.

We remark that such a non-degenerate point $(\mathbf{w},\mathbf{m})$ can be explicitly constructed from a given representation ρ. See Examples 4.3.1 and 4.3.2. We also stress that the assumption on $\mathrm{SL}(2,\mathbb{C})$-lifting does not restrict too many cases. For instance, if $\mathrm{tr}(\rho(\mu_i)) \neq 0$ for all $1 \leqslant i \leqslant h$, then ρ admits a lifting. In particular, any boundary parabolic representation has a lifting. Also, if L is a knot, then any representation $\rho : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ admits a lifting.

For $\kappa = (r_1, s_1, \cdots, r_h, s_h)$ we denote by N_κ the manifold obtained from N by Dehn-filling that kills the curve $r_j\mu_j + s_j\lambda_j$ on each boundary torus where

(r_j, s_j) is either a pair of coprime integers or the symbol ∞ meaning that we do not fill Σ_j.

Let $\rho : \pi_1(N_\kappa) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation. If N_κ has a cusp, we assume that ρ is boundary parabolic so that the complex volume of ρ are well-defined. Regarding ρ as a representation from $\pi_1(N)$ by compositing the inclusion $\pi_1(N) \to \pi_1(N_\kappa)$, we have

$$\begin{cases} \mathrm{tr}(\rho(\mu_i)) = \pm 2,\ \mathrm{tr}(\rho(\lambda_i)) = \pm 2 & \text{for } (r_i, s_i) = \infty \\ \rho(\mu_i^{r_i}\lambda_i^{s_i}) = \pm I & \text{for } (r_i, s_i) \neq \infty. \end{cases} \tag{1.2.4}$$

If we assume that $\rho : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ admits a $\mathrm{SL}(2,\mathbb{C})$-lifting and $\rho(\mu_i) \neq \pm I$ for all $1 \leqslant i \leqslant h$, then by Theorems 1.2.1 and 1.2.2 there exists a non-degenerate point $(\mathbf{w}, \mathbf{m})$ such that $\rho_{\mathbf{w},\mathbf{m}} = \rho$ up to conjugation where m_i is an eigenvalue of $\rho(\mu_i)$. It follows from the equation (1.2.4) that for $(r_i, s_i) \neq \infty$ we have $m_i^{r_i} l_i^{s_i} = \pm 1$ and thus $r_i \log m_i + s_i \log l_i \equiv 0$ in modulo πi where l_i is an eigenvalue of $\rho(\lambda_i)$. From coprimeness of the pair (r_i, s_i), there are integers u_i and v_i satisfying

$$r_i \log m_i + s_i \log l_i + \pi i(r_i u_i + s_i v_i) = 0.$$

Theorem 1.2.3. The complex volume of $\rho : \pi_1(N_\kappa) \to \mathrm{PSL}(2,\mathbb{C})$ is given by

$$i\mathrm{Vol}_{\mathbb{C}}(\rho) \equiv \mathbb{W}_0(\mathbf{w}, \mathbf{m}) \mod \pi^2\mathbb{Z}$$

where the function $\mathbb{W}_0(w_1, \cdots, w_n, m_1, \cdots, m_h)$ is defined by

$$\begin{aligned} \mathbb{W}_0 := \mathbb{W}(w_1, \cdots, w_n, m_1, \cdots, m_h) - \sum_{j=1}^{n} \left(w_j \frac{\partial \mathbb{W}}{\partial w_j} \right) \log w_j \\ - \sum_{(r_i,s_i)\neq\infty} \left[\left(m_i \frac{\partial \mathbb{W}}{\partial m_i} \right) (\log m_i + u_i \pi i) - \frac{r_i}{s_i} (\log m_i + u_i \pi i)^2 \right]. \end{aligned}$$

1.3 Cluster variables

Let D be a braid of length n and width m. Hikami and Inoue [HI15] considered $n+1$ cluster variables $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{n+1}$, each of which consists of $3m+1$ variables, and related two consecutive cluster variables $\mathbf{x}^i$ and $\mathbf{x}^{i+1}$ $(1 \leqslant i \leqslant n)$ by an operator arising from cluster mutations. Precisely, if D has a braid group presentation $\sigma_{k_1}^{\epsilon_1}\sigma_{k_2}^{\epsilon_2}\cdots\sigma_{k_n}^{\epsilon_n}$, where σ_{k_i} denotes the standard generator of the m-braid group and $\epsilon_i = \pm 1$, then we have

$$\mathbf{x}^2 = R_{k_1}^{\epsilon_1}(\mathbf{x}^1),\ \mathbf{x}^3 = R_{k_2}^{\epsilon_2}(\mathbf{x}^2),\ \cdots,\ \mathbf{x}^{n+1} = R_{k_n}^{\epsilon_n}(\mathbf{x}^n).$$

We refer to [HI15] for details.

Definition 1.3.1. The initial cluster variable $\mathbf{x}^1 \in \mathbb{C}^{3m+1}$ is called a *solution* if $\mathbf{x}^1 = \mathbf{x}^{n+1}$.

Recall that the space $S^3\backslash(K \cup \{p, q\})$ admits a decomposition into ideal octahedra, where K is the knot represented by D and $p \neq q \in S^3$ are two points not in K. See, for instance, [Thu99], [Wee05], or Section 5.1.1. Dividing each ideal octahedron into four ideal tetrahedra (as in Figure 4 of [HI15]), Hikami and Inoue proved that a non-degenerate solution (see Definition 5.1.1) determines the shape parameter of each ideal tetrahedron so that these tetrahedra satisfy the gluing equations and completeness condition. In particular, we obtain a boundary-parabolic representation

$$\rho_{\mathbf{x}^1} : \pi_1(S^3\backslash K) = \pi_1(S^3\backslash(K \cup \{p, q\})) \to \mathrm{PSL}(2, \mathbb{C})$$

up to conjugation from a non-degenerate solution $\mathbf{x}^1$.

Conjecture 1.3.1. [HI15, Conjecture 3.2] Let D be a braid presentation of a hyperbolic knot K. Then there exists a non-degenerate solution $\mathbf{x}^1$ such that the induced representation $\rho_{\mathbf{x}^1}$ is geometric, i.e., discrete and faithful.

Remark 1.3.1. In this thesis, we shall divide an ideal octahedron into five tetrahedra, rather than four (see Figure 5.2). A non-degenerate solution, implying the non-degeneracy of the ideal tetrahedra, thus requires a slightly different condition (see Definition 5.1.1) from [HI15]. Henceforth, by a non-degenerate solution we mean a solution that satisfies the condition in Definition 5.1.1. We stress that this change of an ideal triangulation is essential for the existence of a non-degenerate solution (see Remark 5.2.1).

The main purpose of Chapter 5 is to analyze the above conjecture. In particular, we prove the following, which is a consequence of the more general results Theorems 1.3.2 and 1.3.3 below.

Theorem 1.3.1. Conjecture 1.3.1 holds if and only if the length of the braid is odd.

Note that one can always make the braid length odd by adding a kink if necessary.

1.3.1 Overview

Let M be a compact 3-manifold with non-empty boundary and G be either $\mathrm{PSL}(2,\mathbb{C})$ or $\mathrm{SL}(2,\mathbb{C})$. Recall that a representation $\rho : \pi_1(M) \to G$ is *boundary-parabolic* if it maps peripheral subgroups to conjugates of the subgroup P of G consisting of upper triangular matrices with ones on the diagonal. We shall sometimes call such ρ a *(G,P)-representation.*

A representation $\pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ may or may not lift to $\mathrm{SL}(2,\mathbb{C})$ and the obstruction to lifting is a class in $H^2(M;\{\pm 1\})$. Also, a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$-representation may lift to an $\mathrm{SL}(2,\mathbb{C})$-representation which is not boundary-parabolic. The obstruction to lifting a $(\mathrm{PSL}(2,\mathbb{C}),P)$-representation ρ to a $(\mathrm{SL}(2,\mathbb{C}),P)$-representation is a class, called the *obstruction class of ρ*, in $H^2(M,\partial M;\{\pm 1\})$ [GTZ15, GGZ15]. Note that the image of this class in $H^2(M;\{\pm 1\})$ is the obstruction to lifting ρ to $\mathrm{SL}(2,\mathbb{C})$. If $M = S^3 \backslash \nu(K)$, where

$\nu(K)$ denotes a small open regular neighborhood of a knot K, then we have $H^2(M, \partial M; \{\pm 1\}) \simeq \{\pm 1\}$. Therefore, the obstruction class of a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ can be viewed as an element of $\{\pm 1\}$.

Theorem 1.3.2. Let D be a braid of a knot K (not necessarily hyperbolic). Then the obstruction class of $\rho_{\mathbf{x}^1}$ induced from a non-degenerate solution $\mathbf{x}^1$ is $(-1)^n$ where n is the length of D.

The obstruction class of the geometric representation of a hyperbolic knot is non-trivial. This follows from the fact that any lift of the geometric representation maps a longitude to an element with trace -2 (see e.g. [Cal06], [MFP$^+$12, §3.2] and also Proposition 2.2.1 below). Hence, Theorem 1.3.2 shows that having odd braid length is necessary for Conjecture 1.3.1 to hold. The fact that this is also sufficient follows from the result below, which is proved in Section 5.2.2.

Theorem 1.3.3. Let D be a braid of a knot K (not necessarily hyperbolic) and $\rho : \pi_1(S^3 \backslash K) \to \mathrm{PSL}(2,\mathbb{C})$ be a non-trivial boundary-parabolic representation. If the obstruction class of ρ is $(-1)^n$, where n is the length of D, then there exists a non-degenerate solution $\mathbf{x}^1$ such that the induced representation $\rho_{\mathbf{x}^1}$ coincides with ρ up to conjugation.

We remark that the solution can be constructed explicitly when ρ is given using the Wirtinger presentation of the knot group. This uses techniques developed in [Cho16a].

Chapter 2

Preliminaries

2.1 Cocycles

Let X be a topological space equipped with a polyhedral decomposition. We denote by X^i the set of oriented i-cells (unoriented when $i = 0$). For an oriented 1-cell $e \in X^1$ we denote by $-e$ the same edge e with its opposite orientation.

Let G be a group. The set $C^i(X; G)$ of all set maps from X^i to G forms a group with the operation naturally induced from G. We call $\sigma \in C^1(X; G)$ a *cocycle* if (i) $\sigma(e)\sigma(-e) = 1$ for all $e \in X^1$; (ii) $\sigma(e_1)\sigma(e_2)\cdots\sigma(e_m) = 1$ for each face f of X where $e_1, \cdots, e_m$ are the boundary edges of the face in the cyclic order determined by a choice of orientation of f. We denote by $Z^1(X; G)$ the set of all cocycles.

The group $C^0(X; G)$ acts on $Z^1(X; G)$ as follows:

$$Z^1(X; G) \times C^0(X; G) \to Z^1(X; G), \quad (\sigma, \tau) \mapsto \sigma^\tau$$

where $\sigma^\tau : X^1 \to G$ is given by $\sigma^\tau(e) = \tau(v)^{-1}\sigma(e)\tau(w)$ for $e \in X^1$, where v and w are the initial and terminal vertices of e, respectively. The following fact is well-known (see e.g. [Zic09, Neu04]).

Proposition 2.1.1. The orbit space $H^1(X;G) := Z^1(X;G)/C^0(X;G)$ has a natural bijection with the set of all conjugacy classes of representations $\pi_1(X) \to G$.

Note that if G is abelian, $H^1(M;G)$ is canonically isomorphic to the usual cellular cohomology group with the coefficient G.

2.2 Obstruction classes

Let N be an oriented compact 3-manifold with non-empty boundary. We fix an ideal triangulation of the interior of N. This endows N with a decomposition into truncated tetrahedra whose triangular faces triangulate ∂N. A *truncated tetrahedron* is a polyhedron obtained from a tetrahedron by chopping off a small neighborhood of each vertex. We denote by N^i and ∂N^i the set of the oriented i-cells (unoriented when $i = 0$) of N and ∂N, respectively. We call an edge of ∂N a *short edge* and call an edge of N not in ∂N a *long edge*; see Figure 2.1.

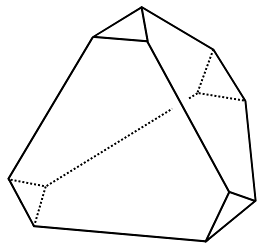


Figure 2.1: A truncated tetrahedron

Let G be either $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$ and P be the subgroup of G consisting of upper triangular matrices with ones on the diagonal. We denote by $C^i(N,\partial N;G,P)$ the subset of $C^i(N;G)$ consisting of $\sigma \in C^i(N;G)$ satisfying $\sigma(x) \in P$ for all $x \in \partial N^i$. We let

$$Z^1(N,\partial N;G,P) = Z^1(N;G) \cap C^1(N,\partial N;G,P),$$
$$H^1(N,\partial N;G,P) = Z^1(N,\partial M;G,P)/C^0(N,\partial N;G,P).$$

An element of $Z^1(N, \partial N; G, P)$ is called a (G,P)-*cocycle*. One can easily check that every (G,P)-representation (see Definition 2.2.1 below) can be represented by a (G,P)-cocycle. We refer to [Zic09, GTZ15] for details.

Definition 2.2.1. A representation $\rho : \pi_1(N) \to G$ is called a (G,P)-*representation* if it maps $\pi_1(\Sigma)$ to the conjugates of P for every component Σ of ∂N.

From the central extension $1 \to \{\pm 1\} \to \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C}) \to 1$, we obtain exact sequences (the standard proof of exactness still works in low degree even though the terms are only sets, not groups)

$$H^1(N; \mathrm{SL}(2,\mathbb{C})) \to H^1(N; \mathrm{PSL}(2,\mathbb{C})) \to H^2(N; \{\pm 1\}) \quad \text{and}$$

$$H^1(N, \partial N; \mathrm{SL}(2,\mathbb{C}), P) \to H^1(N, \partial N; \mathrm{PSL}(2,\mathbb{C}), P) \xrightarrow{\delta} H^2(N, \partial N; \{\pm 1\}).$$

The latter sequence tells us that a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation ρ admits an $(\mathrm{SL}(2,\mathbb{C}), P)$-lifting if and only if $\delta(\rho) \in H^2(N, \partial N; \{\pm 1\})$ vanishes, where we view ρ as a $(\mathrm{PSL}(2,\mathbb{C}), P)$-cocycle. We refer to $\delta(\rho)$ as the *obstruction class* of ρ. Note that it does not depend on the choice of a $(\mathrm{PSL}(2,\mathbb{C}), P)$-cocycle representing ρ.

When N is a knot exterior in S^3, the obstruction class can be directly computed as follows. Recall that in this case we have $H^2(N; \{\pm 1\}) = 0$ and $H^2(N, \partial N; \{\pm 1\}) \cong \{\pm 1\}$; in particular, any $\mathrm{PSL}(2,\mathbb{C})$-representation admits an $\mathrm{SL}(2,\mathbb{C})$-lifting.

Proposition 2.2.1. Let N be a knot exterior in S^3. Then the obstruction class of a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation ρ, viewed as an element of $H^2(N, \partial N; \{\pm 1\}) \simeq \{\pm 1\}$, coincides with half of $\mathrm{tr}(\widetilde{\rho}(\lambda))$ where $\widetilde{\rho} : \pi_1(N) \to$ is any lifting of ρ and λ is the canonical longitude of the knot.

Proof. Considering any Wirtinger presenation of $\pi_1(N)$, it is easy to check that ρ has only two $\mathrm{SL}(2,\mathbb{C})$-liftings $\widetilde{\rho}_+$ and $\widetilde{\rho}_-$ such that $\mathrm{tr}(\widetilde{\rho}_+(\mu)) = 2$ and $\mathrm{tr}(\widetilde{\rho}_-(\mu)) = -2$, respectively, where μ is a meridian of the knot. Since $\pi_1(\partial N)$ is an abelian group generated by μ and λ, ρ admits an $(\mathrm{SL}(2,\mathbb{C}), P)$-lifting if and only if $\mathrm{tr}(\widetilde{\rho}_+(\lambda)) = 2$. Therefore, by definition the obstruction class $\delta(\rho) \in \{\pm 1\}$ coincides with half of $\mathrm{tr}(\widetilde{\rho}_+(\lambda))$. On the other hand, the canonical longitude λ is in the commutator subgroup of $\pi_1(N)$ and thus it should be expressed in Wirtinger generators of even length. Therefore, we have $\widetilde{\rho}_+(\lambda) = \widetilde{\rho}_-(\lambda)$. $\square$

Chapter 3

Ptolemy varieties

Based on the work of Neumann [Neu04], Zickert [Zic09] gave an efficient formula for computing the complex volume of a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation of a compact 3-manifold with non-empty boundary. In this chapter, we give a brief review on [Neu04] (Section 3.1) and extend the formula of Zickert to an arbitrary $\mathrm{PSL}(2,\mathbb{C})$-representation (Section 3.2). This shall allow us to compute the complex volume of a $\mathrm{PSL}(2,\mathbb{C})$-representation of a closed 3-manifold obtained from Dehn filling.

3.1 Formulas of Neumann

We first recall theorems in [Neu04] that we need for our main theorem of this chapter.

Let N be an oriented compact 3-manifold with non-empty boundary and let $\mathcal{T}$ be an ideal triangulation of the interior of N with n ideal tetrahedra $\Delta_1, \cdots, \Delta_n$. Following [Neu04], we assume that each Δ_j has a vertex-ordering so that these orderings agree on the common faces. We say that Δ_j is *positively oriented* if the orientation of Δ_j induced from the vertex-ordering agrees with the orientation of N; Δ_j is *negatively oriented*, otherwise. We let $\epsilon_j = \pm 1$ according to this orientation of Δ_j.

Recall that an ideal tetrahedron with mutually distinct vertices, say $z_0, z_1, z_2, z_3 \in \partial\overline{\mathbb{H}^3}$, is determined up to isometry by the cross-ratio

$$z = [z_0 : z_1 : z_2 : z_3] := \frac{(z_0 - z_3)(z_1 - z_2)}{(z_0 - z_2)(z_1 - z_3)} \in \mathbb{C}\backslash\{0, 1\}$$

where the cross-ratio at each edge is given by one of $z, z' := \frac{1}{1-z}$, and $z'' := 1 - \frac{1}{z}$. See Figure 3.1 (left).

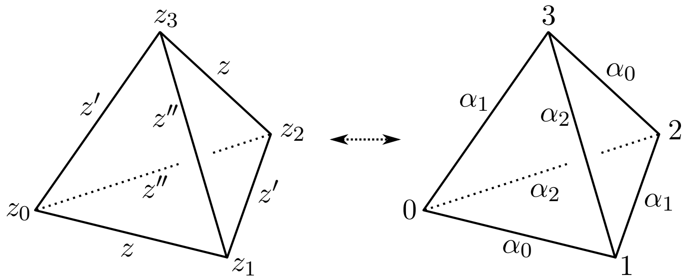


Figure 3.1: Cross-ratios and log-parameters

Definition 3.1.1 ([Neu04])**.** A *flattening* of an ideal tetrahedron with the shape parameter $z \in \mathbb{C}\backslash\{0, 1\}$ is a triple $\alpha = (\alpha^0, \alpha^1, \alpha^2) \in \mathbb{C}^3$ of the form

$$\begin{cases} \alpha^0 &= \log z + p\pi i \\ \alpha^1 &= -\log(1-z) + q\pi i \\ \alpha^2 &= -\log z + \log(1-z) - (p+q)\pi i \end{cases}$$

for some $p, q \in \mathbb{Z}$. Alternatively, a *flattening* is a triple $\alpha = (\alpha^0, \alpha^1, \alpha^2) \in \mathbb{C}^3$ satisfying $\alpha^0 + \alpha^1 + \alpha^2 = 0$ and

$$\alpha^0 \equiv \log z,\ \alpha^1 \equiv \log z',\ \alpha^2 \equiv \log z''$$

in modulo πi.

We refer to the complex numbers α^0, α^1, and α^2 as *log-parameters* and assign each of them to an edge accordingly as in Figure 3.1. Remark that a flattening $\alpha = (\alpha^0, \alpha^1, \alpha^2)$ determines and is determined by another triple $(z; p, q)$ (see [Neu04, Lemma 3.2]). We thus often write the flattening α in either way: $(\alpha^0, \alpha^1, \alpha^2)$ or $(z; p, q)$.

A closed path in the interior of N is called a *normal path* if it meets no edges of any Δ_j and crosses faces only transversally. When a normal path passes through Δ_j, we may assume that up to homotopy it enters and departs at different faces of Δ_j so that there is a unique edge of Δ_j between these faces. See, for instance, Figures 3.7 and 3.9. We say that the path *passes* this edge. By *the sum of log-parameters along a normal path*, we mean the signed-sum of log-parameters over all edges that the path passes. We refer to [Neu04] for the signed-sum convention. In particular, when a normal path winds an edge of $\mathcal{T}$ as in Figure 3.7, such a sum is called *the sum of log-parameters around the edge*.

Theorem 3.1.1 ([Neu04])**.** Suppose that the interior of N decomposes into n ideal tetrahedra $\Delta_1, \cdots, \Delta_n$. Then for any collection of flattenings α_j of Δ_j satisfying

- parity condition : parity along each normal path is zero;
- edge condition : the sum of log-parameters around each edge of $\mathcal{T}$ is zero;
- cusp condition : the sum of log-parameters along any normal path in the neighborhood of an ideal vertex of $\mathcal{T}$ is zero,

we obtain

$$i\mathrm{Vol}_{\mathbb{C}}(\rho) \equiv \sum_{j=1}^{n} \epsilon_j R(\alpha_j) \mod \pi^2\mathbb{Z}$$

where $\rho : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ is a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation induced from the flattenings and R denotes the extended Rogers dilogarithm;

$$R(z; p, q) = \mathrm{Li}_2(z) + \frac{\pi i}{2}(p \log(1-z) + q \log z) + \frac{1}{2}\log(1-z)\log z - \frac{\pi^2}{2}.$$

Theorem 3.1.1 extends to a Dehn-filled manifold as Theorem 3.1.2 below. We denote the components of ∂N by $\Sigma_1, \cdots, \Sigma_h$ and assume that each component Σ_j is a torus with a fixed meridian μ_j and longitude λ_j. For $\kappa = (r_1, s_1, \cdots, r_h, s_h)$ we denote by N_κ the manifold obtained from N by performing the Dehn filling that kills the curve $r_j\mu_j + s_j\lambda_j$ on each Σ_j, where (r_j, s_j) is either a pair of coprime integers or the symbol ∞ meaning that we do not fill Σ_j.

Theorem 3.1.2. [Neu04, Theorem 14.7] Let N_κ be a Dehn-filled manifold obtained from N. Then for any collection of flattenings α_j of Δ_j satisfying

- parity condition : parity along each normal path is zero;
- edge condition : the sum of log-parameters around each edge of $\mathcal{T}$ is zero;
- cusp condition : the sum of log-parameters along any normal path in the neighborhood of an ideal vertex of $\mathcal{T}$ that represents an unfilled cusp is zero;
- Dehn-filling condition : the sum of log-parameters along any normal path in the neighborhood of an ideal vertex of $\mathcal{T}$ that represents a filled cusp is zero if the path is null-homotopic in the added torus,

we obtain the induced representation $\rho : \pi_1(N_\kappa) \to \mathrm{PSL}(2, \mathbb{C})$ and

$$i\mathrm{Vol}_{\mathbb{C}}(\rho) \equiv \sum_{j=1}^{n} \epsilon_j R(\alpha_j) \mod \pi^2\mathbb{Z}. \tag{3.1.1}$$

3.2 Deformed Ptolemy varieties

Let N be an oriented compact 3-manifold with non-empty boundary. Let $\mathcal{T}$ be an ideal triangulation of the interior of N. Recall that this endows N with a decomposition into truncated tetrahedra whose triangular faces triangulate

∂N (see Figure 2.1). We denote by X^1 the set of oriented 1-cells of X where $X = \partial N, N$, and $\mathcal{T}$. An edge $e \in N^1$ is called a *short-edge* if $e \in \partial N^1$; a *long-edge* otherwise. We shall confuse an edge $e \in \mathcal{T}^1$ with a long-edge of N in a natural way.

A cocycle $\phi \in Z^1(N; \mathrm{SL}(2,\mathbb{C}))$ is called a *natural cocycle* if $\phi(e)$ is of the counter-diagonal form for all long-edges e and is of the upper-triangular form for all short-edges e. Note that the term 'natural' is followed from [GTZ15, GGZ15]. A natural cocycle ϕ corresponds to a pair of assignments $\sigma : \partial N^1 \to \mathbb{C}^\times$ and $c : N^1 \to \mathbb{C}$ satisfying

$$\begin{cases} \phi(e) = \begin{pmatrix} 0 & -c(e)^{-1} \\ c(e) & 0 \end{pmatrix} & \text{for all long-edges } e; \\ \phi(e) = \begin{pmatrix} \sigma(e) & c(e) \\ 0 & \sigma(e)^{-1} \end{pmatrix} & \text{for all short-edges } e. \end{cases} \tag{3.2.2}$$

We call $c(e)$ a *short edge parameter* or a *long-edge parameter* according to an edge e. Note that (i) $c(-e) = -c(e)$ for all $e \in N^1$; (ii) each long-edge parameter is non-zero; (iii) the assignment $\sigma : \partial N^1 \to \mathbb{C}^\times$ should be a cocycle, regarding $\mathbb{C}^\times$ as the multiplicative group. We refer to σ as the *boundary cocycle* of ϕ.

Proposition 3.2.1. We consider a hexagonal face of N and denote the edges as in Figure 3.2. Then ϕ satisfies cocycle condition for the face if and only if

$$\begin{cases} c(s_{12}) = -\dfrac{\sigma(s_{31})}{\sigma(s_{23})} \dfrac{c(l_3)}{c(l_1)c(l_2)}; \\ c(s_{23}) = -\dfrac{\sigma(s_{12})}{\sigma(s_{31})} \dfrac{c(l_1)}{c(l_2)c(l_3)}; \\ c(s_{31}) = -\dfrac{\sigma(s_{23})}{\sigma(s_{12})} \dfrac{c(l_2)}{c(l_3)c(l_1)}. \end{cases} \tag{3.2.3}$$

Proof. The cocycle condition $\phi(l_1)\,\phi(s_{12})\,\phi(l_2)\,\phi(s_{23})\,\phi(l_3)\,\phi(s_{31}) = I$ is equiva-

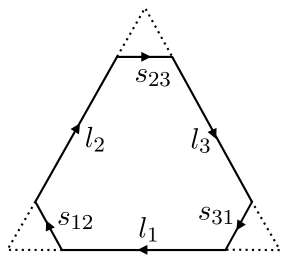


Figure 3.2: A hexagonal face of N

lent to

$$\phi(l_1)\,\phi(s_{12})\,\phi(l_2) = \phi(s_{31})^{-1}\phi(l_3)^{-1}\phi(s_{23})^{-1}$$

$$\Leftrightarrow \begin{pmatrix} -\frac{c(l_2)}{\sigma(s_{12})c(l_1)} & 0 \\ c(l_1)c(l_2)c(s_{12}) & -\frac{\sigma(s_{12})c(l_1)}{c(l_2)} \end{pmatrix} = \begin{pmatrix} \frac{c(l_3)c(s_{31})}{\sigma(s_{23})} & -c(l_3)c(s_{23})c(s_{31}) + \frac{\sigma(s_{23})}{\sigma(s_{31})c(l_3)} \\ -\frac{\sigma(s_{31})c(l_3)}{\sigma(s_{23})} & \sigma(s_{31})c(s_{23})c(l_3) \end{pmatrix}.$$

We directly obtain the equation (3.2.3) by comparing the entries of the above two matrices. □

The above proposition tells us that every short-edge parameter is non-zero and is uniquely determined by the boundary cocycle σ and long-edge parameters.

Proposition 3.2.2. We consider a truncated tetrahedron of N and denote the long-edges as in Figure 3.3. We also denote by s_{ij} the short-edge running from l_i to l_j as in Figure 3.3. Then ϕ satisfies cocycle condition for all triangular faces on its boundary if and only if

$$c(l_3)c(l_6) = \frac{\sigma(s_{23})}{\sigma(s_{35})}\frac{\sigma(s_{26})}{\sigma(s_{65})}c(l_2)c(l_5) + \frac{\sigma(s_{13})}{\sigma(s_{34})}\frac{\sigma(s_{16})}{\sigma(s_{64})}c(l_1)c(l_4). \tag{3.2.4}$$

We call the equation (3.2.4) the *σ-deformed Ptolemy equation.*

Proof. The cocycle condition $\phi(s_{23})\phi(s_{34}) = \phi(s_{24})$ for the top triangular face is equivalent to $c(s_{24}) = \sigma(s_{23})c(s_{34}) + \sigma(s_{34})^{-1}c(s_{23})$. Replacing three short-edge

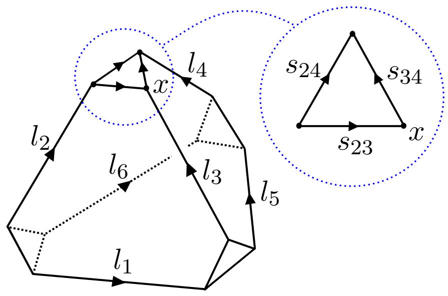


Figure 3.3: A truncated tetrahedron of N

parameters $c(s_{34}), c(s_{23})$, and $c(s_{24})$ by σ and $c(l_i)$ through Proposition 3.2.1, we obtain the equation (3.2.4):

$$\begin{aligned} c(s_{24}) =& \sigma(s_{23})c(s_{34}) + \sigma(s_{34})^{-1}c(s_{23}) \\ \Leftrightarrow -\frac{\sigma(s_{62})}{\sigma(s_{46})}\frac{c(l_6)}{c(l_2)c(l_4)} =& -\sigma(s_{23})\frac{\sigma(s_{53})}{\sigma(s_{45})}\frac{c(l_5)}{c(l_3)c(l_4)} - \sigma(s_{34})^{-1}\frac{\sigma(s_{12})}{\sigma(s_{31})}\frac{c(l_1)}{c(l_2)c(l_3)} \\ \Leftrightarrow c(l_3)c(l_6) =& \frac{\sigma(s_{23})}{\sigma(s_{35})}\frac{\sigma(s_{26})}{\sigma(s_{65})}c(l_2)c(l_5) + \frac{\sigma(s_{13})}{\sigma(s_{34})}\frac{\sigma(s_{16})}{\sigma(s_{64})}c(l_1)c(l_4). \end{aligned}$$

We compute similarly for other three triangular faces, each of which results in the same equation (3.2.4). $\square$

Definition 3.2.1. The *σ-deformed Ptolemy variety*, denoted by $P_\sigma(\mathcal{T})$, for $\sigma \in Z^1(\partial N; \mathbb{C}^\times)$ is the set of all assignments $c : \mathcal{T}^1 \to \mathbb{C}^\times$ satisfying $-c(e) = c(-e)$ for all $e \in \mathcal{T}^1$ and the *σ-deformed Ptolemy equation* (3.2.4) for each ideal tetrahedron of $\mathcal{T}$.

Propositions 3.2.1 and 3.2.2 tell us that the equation (3.2.2) gives the one-to-one correspondence

$$\coprod_{\sigma \in Z^1(\partial N; \mathbb{C}^\times)} P_\sigma(\mathcal{T}) \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{c} \text{natural cocycles} \\ \phi \in Z^1(N; \mathrm{SL}(2, \mathbb{C})) \end{array} \right\} \tag{3.2.5}$$

In particular, $P_\sigma(\mathcal{T})$ corresponds to the set of all natural cocycles whose boundary cocycle is σ.

Remark 3.2.1. When σ is trivial, i.e. $\sigma(e) = 1$ for all $e \in \partial N^1$, the σ-deformed Ptolemy variety $P_\sigma(\mathcal{T})$ reduces to the Ptolemy variety defined in [GTZ15]. This interprets $P_\sigma(\mathcal{T})$ as a generalization of the Ptolemy variety.

Recall that any (natural) cocycle determines a $\mathrm{SL}(2,\mathbb{C})$-representation of $\pi_1(N)$ uniquely up to conjugation. We thus obtain the set map

$$\rho : \coprod_\sigma P_\sigma(\mathcal{T}) \to \mathrm{Hom}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))/_{\mathrm{Conj}}, \; c \mapsto \rho_c.$$

For each component, say Σ, of ∂N it follows from the equation (3.2.2) that

$$\rho_c(\gamma) = \begin{pmatrix} \sigma_\Sigma(\gamma) & * \\ 0 & \sigma_\Sigma(\gamma)^{-1} \end{pmatrix} \tag{3.2.6}$$

up to conjugation for all $\gamma \in \pi_1(\Sigma)$. Note that one can discard conjugation ambiguity of ρ_c by fixing a base point of $\pi_1(N)$, while the homomorphism $\sigma_\Sigma : \pi_1(\Sigma) \to \mathbb{C}^\times$ has no conjugation ambiguity from the first (since the group $\mathbb{C}^\times$ is commutative).

3.2.1 Isomorphisms

Recall that two cocycles $\sigma, \sigma' \in Z^1(\partial N; \mathbb{C}^\times)$ give the same homomorphism on each component of ∂N if and only if $\sigma' = \sigma^\tau$ for some $\tau \in C^0(\partial N; \mathbb{C}^\times)$. In the case, we define a map

$$\Phi : P_\sigma(\mathcal{T}) \to P_{\sigma^\tau}(\mathcal{T}), \; c \mapsto c^\tau$$

by $c^\tau(e) = \tau(v_1)\,\tau(v_2)\,c(e)$ for all $e \in \mathcal{T}^1$ where v_1 and $v_2 \in N^0$ are the endpoints of e, viewed as a long-edge of N.

Proposition 3.2.3. Φ is a well-defined isomorphism.

Proof. Note that $\sigma^{\tau_1\tau_2} = (\sigma^{\tau_1})^{\tau_2}$ for any $\tau_1, \tau_2 \in C^0(\partial N; \mathbb{C}^\times)$. We thus may assume that $\tau \in C^0(\partial N; \mathbb{C}^\times)$ is trivial except on a single vertex $x \in \partial N^0$. Suppose x is the initial vertex of the long-edge l_3 as in Figure 3.3. Then, in the equation (3.2.4), only two terms $\sigma(s_{23})$ and $\sigma(s_{34})$ are affected by the τ-action: $\sigma^\tau(s_{23}) = \sigma(s_{23})\tau(x)$ and $\sigma^\tau(s_{34}) = \tau(x)^{-1}\sigma(s_{34})$. Multiplying $\tau(x)$ to both sides of the equation (3.2.4), we have $c^\tau \in P_{\sigma^\tau}(\mathcal{T})$:

$$\tau(x)c(l_3)\,c(l_6) = \frac{\sigma(s_{23})\tau(x)}{\sigma(s_{35})}\frac{\sigma(s_{26})}{\sigma(s_{65})}c(l_2)c(l_5) + \frac{\sigma(s_{13})}{\tau(x)^{-1}\sigma(s_{34})}\frac{\sigma(s_{16})}{\sigma(s_{64})}c(l_1)c(l_4)$$

$$\Leftrightarrow c^\tau(l_3)\,c^\tau(l_6) = \frac{\sigma^\tau(s_{23})}{\sigma^\tau(s_{35})}\frac{\sigma^\tau(s_{26})}{\sigma^\tau(s_{65})}c^\tau(l_2)c^\tau(l_5) + \frac{\sigma^\tau(s_{13})}{\sigma^\tau(s_{34})}\frac{\sigma^\tau(s_{16})}{\sigma^\tau(s_{64})}c^\tau(l_1)c^\tau(l_4).$$

Recall that $c^\tau(l_i) = c(l_i)$ for $i \in \{1,2,4,5,6\}$ and $c^\tau(l_3) = \tau(x)c(l_3)$. On the other hand, the inverse $\tau^{-1} \in C^0(\partial N; \mathbb{C}^\times)$ (as a group element) exactly gives the inverse morphism of Φ. $\square$

Proposition 3.2.4. The following diagram commutes:

$$\begin{tikzcd}
P_\sigma(\mathcal{T}) \arrow[d, "\Phi", "\simeq"'] \arrow[dr, "\rho"] & \\
P_{\sigma^\tau}(\mathcal{T}) \arrow[r, "\rho"] & \mathrm{Hom}(\pi_1(N),\, \mathrm{SL}(2,\mathbb{C}))/_{\mathrm{Conj}}
\end{tikzcd}$$

Proof. Let ϕ_c and $\phi_{c^\tau} \in Z^1(N; \mathrm{SL}(2,\mathbb{C}))$ be the natural cocycles corresponding to $c \in P_\sigma(\mathcal{T})$ and $\Phi(c) = c^\tau \in P_{\sigma^\tau}(\mathcal{T})$, respectively. Let $\hat{\tau} \in C^0(N; \mathrm{SL}(2,\mathbb{C}))$ be an assignment given by

$$\hat{\tau}(v) = \begin{pmatrix} \tau(v) & 0 \\ 0 & \tau(v)^{-1} \end{pmatrix}$$

for all $v \in N^0 = \partial N^0$. As in the proof of Proposition 3.2.3, we may assume that $\hat{\tau}$ is trivial except at the single vertex x as in Figure 3.3. The following equations show that $\phi_{c^\tau} = (\phi_c)^{\hat{\tau}}$:

$$\text{at } l_3 : \begin{pmatrix} 0 & -\tau(x)^{-1}c(l_3)^{-1} \\ \tau(x)c(l_3) & 0 \end{pmatrix} = \begin{pmatrix} 0 & c(l_3)^{-1} \\ c(l_3) & 0 \end{pmatrix} \hat{\tau}(x)$$

$$\text{at } s_{23} : \begin{pmatrix} \tau(x)\sigma(s_{23}) & \frac{\sigma(s_{31})}{\sigma(s_{12})}\frac{c(l_1)}{\tau(x)c(l_3)c(l_2)} \\ 0 & \tau(x)^{-1}\sigma(s_{23})^{-1} \end{pmatrix} = \begin{pmatrix} \sigma(s_{23}) & \frac{\sigma(s_{31})}{\sigma(s_{12})}\frac{c(l_1)}{c(l_3)c(l_2)} \\ 0 & \sigma(s_{23})^{-1} \end{pmatrix} \hat{\tau}(x)$$

$$\text{at } s_{34} : \begin{pmatrix} \tau(x)^{-1}\sigma(s_{34}) & \frac{\sigma(s_{45})}{\sigma(s_{53})}\frac{c(l_5)}{\tau(x)c(l_4)c(l_3)} \\ 0 & \tau(x)\sigma(s_{34})^{-1} \end{pmatrix} = \hat{\tau}(x)^{-1} \begin{pmatrix} \sigma(s_{34}) & \frac{\sigma(s_{45})}{\sigma(s_{53})}\frac{c(l_5)}{c(l_4)c(l_3)} \\ 0 & \sigma(s_{34})^{-1} \end{pmatrix}$$

for Figure 3.3. Therefore, the induced homomorphisms ρ_c and ρ_{c^τ} agree up to conjugation. □

The cocycle σ^τ coincides with σ if and only if $\tau \in C^0(\partial N; \mathbb{C}^\times)$ is constant on each component of ∂N. In this case, the map Φ induces a $(\mathbb{C}^\times)^h$-action on $P_\sigma(\mathcal{T})$, called the *diagonal action* [GGZ15, Zic16], where h is the number of the components of ∂N. Precisely, enumerating the components of ∂N by $\Sigma_1, \cdots, \Sigma_h$, we have

$$(\mathbb{C}^\times)^h \times P_\sigma(\mathcal{T}) \to P_\sigma(\mathcal{T}), \quad ((z_1, \cdots, z_h), c) \mapsto (z_1, \cdots, z_h) \cdot c,$$

where $(z_1, \cdots, z_h) \cdot c : \mathcal{T}^1 \to \mathbb{C}^\times$ is defined by $((z_1, \cdots, z_h) \cdot c)(e) = z_i z_j c(e)$ where i and j (possibly $i = j$) are the indices of the components of ∂N joined by e.

Definition 3.2.2. The *reduced σ-deformed Ptolemy variety* $\overline{P}_\sigma(\mathcal{T})$ is the quotient of $P_\sigma(\mathcal{T})$ by the diagonal action.

Example 3.2.1. Let N be the knot exterior of the figure-eight knot in S^3. It is known that the interior of N can be decomposed into two ideal tetrahedra Δ_1 and Δ_2 [Thu78]. We denote the long edges by l_1 and l_2, and the short-edges by $s_1, s_2, \cdots, s_{12}$ as in Figure 3.4. We choose a meridian μ and a longitude λ of the knot as in Figure 3.5. Note that the longitude λ here is inversed to the one in [Thu78].

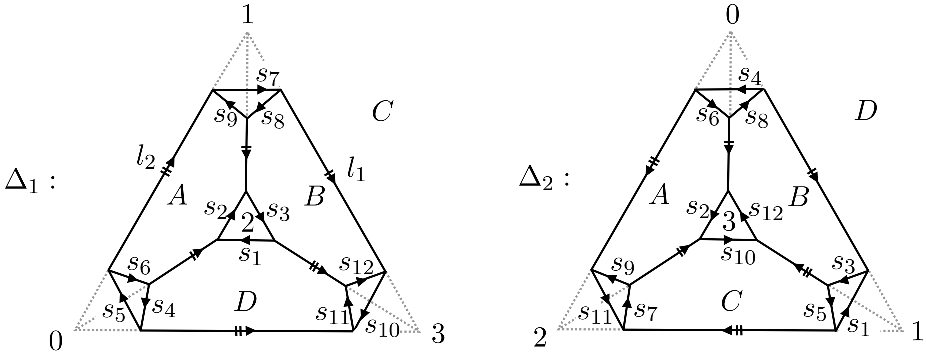


Figure 3.4: The figure eight knot complement

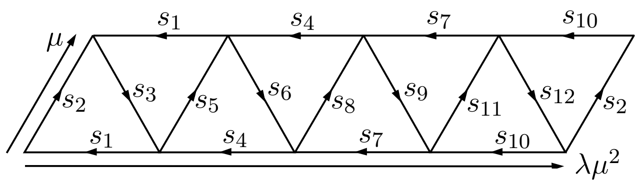


Figure 3.5: The boundary torus

Let Σ be the boundary torus of N. We choose a boundary cocycle $\sigma \in Z^1(\Sigma; \mathbb{C}^\times)$ for $M, L \in \mathbb{C}^\times$ as follows so that the induced homomorphism $\sigma_\Sigma : \pi_1(\Sigma) \to \mathbb{C}^\times$ satisfies $\sigma_\Sigma(\mu) = M$ and $\sigma_\Sigma(\lambda) = L$: $\sigma(s_4) = \sigma(s_7) = \sigma(s_{10}) = 1$, $\sigma(s_2) = \sigma(s_5) = \sigma(s_8) = \sigma(s_{11}) = M$, $\sigma(s_6) = \sigma(s_9) = \sigma(s_{12}) = M^{-1}$, $\sigma(s_1) = L^{-1}M^{-2}$, and $\sigma(s_3) = LM$.

The σ-deformed Ptolemy variety $P_\sigma(\mathcal{T})$ is given by the set of all assignments $c : \{l_1, l_2\} \to \mathbb{C}^\times$ satisfying

$$\begin{cases} \Delta_1 : & -c(l_1)c(l_2) &= & L^{-1}M^{-2}\, c(l_2)^2 - M^2\, c(l_1)^2 \\ \Delta_2 : & c(l_1)c(l_2) &= & c(l_2)^2 - L\, c(l_1)^2 \end{cases}$$

(with $c(-l_i) = -c(l_i)$). The reduced σ-deformed Ptolemy variety $\overline{P}_\sigma(\mathcal{T})$ can be identified with the set of all $z = \frac{c(l_1)}{c(l_2)} \in \mathbb{C}^\times$ satisfying

$$L^{-1}M^{-2} + z - M^2 z^2 = 0 \text{ and } 1 - z - L\, z^2 = 0.$$

Taking the resultant of these two quadratic equations to eliminate z, we obtain

$$L - LM^2 - M^4 - 2LM^4 - L^2M^4 - LM^6 + LM^8 = 0 \tag{3.2.7}$$

which is the A-polynomial of the figure-eight knot [CCG⁺94]. It is clear that the pair (M, L) should satisfy the equation (3.2.7), otherwise $P_\sigma(\mathcal{T})$ shall be empty.

3.2.2 Pseudo-developing maps

Recall that N is a compact 3-manifold with non-empty boundary and $\mathcal{T}$ is an ideal triangulation of the interior of N. Let $\widetilde{N}$ be the universal cover of N and let $\widehat{N}$ be a topological space obtained from $\widetilde{N}$ by collapsing each boundary component to a point. We call these points the *vertices* of $\widehat{N}$. The lifting of $\mathcal{T}$ to the interior of $\widetilde{N}$ induces the notion of *long edges* and *short edges* of $\widetilde{N}$, and also the notion of *edges* of $\widehat{N}$.

We fix a base point x_0 of $\pi_1(N)$ in N^0 together with its lifting $\widetilde{x}_0$ in $\widetilde{N}^0$ so as to fix the $\pi_1(N)$-action on $\widehat{N}$.

Definition 3.2.3. A pair $(\mathcal{D}, \rho)$ of a map $\mathcal{D} : \widehat{N} \to \overline{\mathbb{H}^3}$ and a representation $\rho : \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$ is called a *pseudo-developing map* if

- $\mathcal{D}$ is ρ-equivariant, i.e. $\mathcal{D}(\gamma \cdot x) = \rho(\gamma)\,\mathcal{D}(x)$ for all $\gamma \in \pi_1(N)$ and $x \in \widehat{N}$;
- $\mathcal{D}$ sends all vertices of $\widehat{N}$ to $\partial\overline{\mathbb{H}^3}$;
- $\mathcal{D}(v_1) \neq \mathcal{D}(v_2)$ for every pair of vertices v_1 and v_2 joined by an edge of $\widehat{N}$.

Note that if $(\mathcal{D}, \rho)$ is a pseudo-developing map, then $(g\mathcal{D}, g\rho g^{-1})$ is also a pseudo-developing map for any $g \in$. We say that two pseudo-developing maps

$(\mathcal{D}_1, \rho_1)$ and $(\mathcal{D}_2, \rho_2)$ are *equivalent* if $\rho_2 = g\rho_1 g^{-1}$ and $\mathcal{D}_2$ coincides with $g\mathcal{D}_1$ only on the vertices of $\widehat{N}$ for some $g \in \mathrm{SL}(2, \mathbb{C})$. We denote the equivalence class of $(\mathcal{D}, \rho)$ by $[\mathcal{D}, \rho]$. We refer to [Zic09] for details.

In this subsection, we clarify a relationship between natural cocycles and pseudo-developing maps. We first construct an intermediate object, called a decoration (cf. [Zic09, Definition 3.1]).

Definition 3.2.4. A pair (ψ, ρ) of an assignment $\psi : \widetilde{N}^0 \to \mathbb{C}^2$ and a representation $\rho : \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$ is called a *decoration* if

- ψ is ρ-equivariant, $\psi(\gamma \cdot v) = \rho(\gamma)\psi(v)$ for all $\gamma \in \pi_1(N)$ and $v \in \widetilde{N}^0$;
- $\det\big(\psi(v_1),\, \psi(v_2)\big) \neq 0$ if v_1 and v_2 are joined by a long-edge of $\widetilde{N}$;
- $\det\big(\psi(v_1),\, \psi(v_2)\big) = 0$ if v_1 and v_2 are joined by a short-edge of $\widetilde{N}$,

where an element of $\mathbb{C}^2$ is viewed as a column vector. Note that the second condition implies that $\psi(v)$ should be non-zero for all $v \in \widetilde{N}^0$.

We first construct a correspondence

$$\left\{ \begin{array}{c} \text{natural cocycles} \\ \phi \in Z^1(N; \mathrm{SL}(2, \mathbb{C})) \end{array} \right\} \to \{\text{decorations } (\psi, \rho)\} /_{\sim} \tag{3.2.8}$$

where the equivalence relation $\sim$ in the right-hand side is defined by $(\psi, \rho) \sim (g\psi, g\rho g^{-1})$ for $g \in \mathrm{SL}(2, \mathbb{C})$. We denote the equivalence class of (ψ, ρ) by $[\psi, \rho]$. Since the base point of $\pi_1(N)$ is fixed, a natural cocycle $\phi \in Z^1(N; \mathrm{SL}(2, \mathbb{C}))$ induces a unique homomorphism $\rho : \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$ without conjugation ambiguity. We denote by $\widetilde{\phi} \in Z^1(\widetilde{N}; \mathrm{SL}(2, \mathbb{C}))$ the cocycle obtained by lifting ϕ. We then consider an assignment $\widetilde{\phi}_V \in C^0(\widetilde{N}; \mathrm{SL}(2, \mathbb{C}))$ satisfying

$$\widetilde{\phi}_V(\widetilde{x}_0) = I \text{ and } \widetilde{\phi}(e) = \widetilde{\phi}_V(v_1)^{-1}\, \widetilde{\phi}_V(v_2) \tag{3.2.9}$$

for all $e \in \widetilde{N}^1$, where v_1 and v_2 denote the initial and terminal vertices of e, respectively. Such an assignment $\widetilde{\phi}_V$ exists uniquely and is by definition ρ-equivariant. Finally, we define $\psi : \widetilde{N}^0 \to \mathbb{C}^2$ by the first column part of $\widetilde{\phi}_V$, i.e.

$$\psi(x) = \widetilde{\phi}_V(x)\begin{pmatrix}1\\0\end{pmatrix}$$

for all $x \in \widetilde{N}^0$. From the facts that $\widetilde{\phi}_V$ is ρ-equivariant and ϕ is a natural cocycle, the pair (ψ, ρ) is a decoration. We define the correspondence (3.2.8) by sending ϕ to $[\psi, \rho]$.

Proposition 3.2.5. The correspondence $\phi \mapsto [\psi, \rho]$ is surjective.

Proof. Let (ψ, ρ) be any decoration. We define $\widetilde{\phi}_V \in C^0(\widetilde{N}; \mathrm{SL}(2, \mathbb{C}))$ by

$$\widetilde{\phi}_V(x) := \left(\psi(x), \quad \tfrac{1}{\det(\psi(x),\psi(x'))}\,\psi(x')\right) \in$$

for all $x \in \widetilde{N}^0$, where x' is another vertex of $\widetilde{N}$ connected with x by a long-edge. The second condition of decoration guarantees $\det(\psi(x), \psi(x')) \neq 0$. Since ψ is ρ-equivariant, so is $\widetilde{\phi}_V$. We define $\widetilde{\phi} \in Z^1(\widetilde{N}; \mathrm{SL}(2, \mathbb{C}))$ by $\widetilde{\phi}(e) = \widetilde{\phi}_V(v_1)^{-1}\,\widetilde{\phi}_V(v_2)$ for all $e \in \widetilde{N}^1$, where v_1 and v_2 denote the initial and terminal vertices of e, respectively. Then it satisfies

$$\begin{aligned}
\widetilde{\phi}(\gamma \cdot e) &= \widetilde{\phi}_V(\gamma \cdot v_1)^{-1}\widetilde{\phi}_V(\gamma \cdot v_2)\\
&= (\rho(\gamma)\widetilde{\phi}_V(v_1))^{-1}\rho(\gamma)\widetilde{\phi}_V(v_2)\\
&= \widetilde{\phi}_V(v_1)^{-1}\widetilde{\phi}_V(v_2)\\
&= \widetilde{\phi}(e)
\end{aligned}$$

for all $\gamma \in \pi_1(N)$ and $e \in \widetilde{N}^1$. Therefore, we obtain $\phi \in Z^1(N; \mathrm{SL}(2, \mathbb{C}))$ by projecting $\widetilde{\phi}$ to N. One can check that ϕ is a natural cocycle and hence the correspondence (3.2.8) is surjective. □

Remark 3.2.2. Let (ψ, ρ) be a decoration and let $c \in P_\sigma(\mathcal{T})$ be a corresponding element under the correspondences (3.2.5) and (3.2.8). Then σ and c can be directly determined by ψ as follows. For an edge $e \in N^1$

$$\begin{cases} \psi(v_2) = \sigma(e)\,\psi(v_1) & \text{if } e \text{ is a short-edge} \\ c(e) = \det\big(\psi(v_1),\, \psi(v_2)\big) & \text{if } e \text{ is a long-edge.} \end{cases}$$

where v_1 and v_2 are the initial and terminal vertices of any lifting of e, respectively. Note that (ψ, ρ) and $(g\psi, g\rho g^{-1})$ determine the same σ and c.

We now construct a pseudo-developing map $(\mathcal{D}, \rho)$ from a decoration (ψ, ρ). For a non-zero $C = (c_1, c_2)^t \in \mathbb{C}^2$ let $h(C) = c_1/c_2 \in \mathbb{C} \cup \{\infty\} = \partial\overline{\mathbb{H}^3}$. We first define a map $\mathcal{D} : \widehat{N} \to \overline{\mathbb{H}^3}$ on each vertex v of $\widehat{N}$ by

$$\mathcal{D}(v) = h\left(\psi(x)\right) \tag{3.2.10}$$

where $x \in \widetilde{N}^0$ is arbitrarily chosen in the link of v. The well-definedness of $\mathcal{D}$ follows from the fact that $h(C_1) = h(C_2)$ if and only if $\det(C_1, C_2) = 0$ for non-zero C_1 and $C_2 \in \mathbb{C}^2$. Also, recall the third condition in the definition of a decoration. Furthermore, the first and second conditions of a decoration guarantee the first and third conditions in Definition 3.2.3, respectively. Now we extend $\mathcal{D}$ over the higher dimensional cells in order. See [CS83, §4.5]. Such an extension is unique up to the equivalence relation. This defines a correspondence

$$\{\text{decorations } (\psi, \rho)\}/_\sim \;\to \left\{ \begin{array}{c} \text{pseudo-developing} \\ \text{maps } (\mathcal{D}, \rho) \end{array} \right\}/_\sim \tag{3.2.11}$$

by sending $[\psi, \rho]$ to $[\mathcal{D}, \rho]$.

Proposition 3.2.6. The above correspondence $[\psi, \rho] \mapsto [\mathcal{D}, \rho]$ is surjective.

Proof. Let $(\mathcal{D}, \rho)$ be a pseudo-developing map. Since $\pi_1(N)$ acts freely on $\widetilde{N}^0$, there exists a ρ-equivariant assignment $\psi : \widetilde{N}^0 \to \mathbb{C}^2$ satisfying the equation (3.2.10) for every pair of a vertex v of $\widehat{N}$ and $x \in \widetilde{N}^0$ contained in the link of v.

Then the pair (ψ, ρ) should be automatically a decoration, so the correspondence (3.2.11) is surjective. $\square$

Summing up all the correspondences (3.2.5), (3.2.8), and (3.2.11), we obtain

$$\coprod_{\sigma} P_\sigma(\mathcal{T}) \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{natural cocycles} \\ \phi \in Z(N; \mathrm{SL}(2,\mathbb{C})) \end{array} \right\}$$
$$\twoheadrightarrow \{\text{decorations } (\psi, \rho)\}/_\sim \twoheadrightarrow \left\{ \begin{array}{c} \text{pseudo-developing} \\ \text{maps } (\mathcal{D}, \rho) \end{array} \right\}/_\sim.$$

Whenever we choose $c \in P_\sigma(\mathcal{T})$, each ideal tetrahedron Δ_j of $\mathcal{T}$ admits a non-degenerated hyperbolic structure.

Proposition 3.2.7. The cross-ratio $r(\Delta_j, l_3)$ of Δ_j at the edge l_3 is

$$r(\Delta_j, l_3) = \frac{\sigma(s_{12})\sigma(s_{45})}{\sigma(s_{24})\sigma(s_{51})} \frac{c(l_1)c(l_4)}{c(l_2)c(l_5)} \tag{3.2.12}$$

where l_i's denote the edges of Δ_j as in Figure 3.6 and s_{ik} denotes the short-edge running from l_i to l_k.

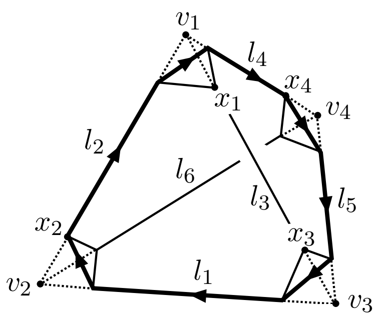


Figure 3.6: An ideal tetrahedron with its truncation

Proof. We choose any lifting of Δ_j in $\widetilde{N}$ and identify it with its developing image. We denote its vertices by $v_1, \cdots, v_4$ as in Figure 3.6. We choose a vertex

$x_i \in \widetilde{N}^0$ in the link of v_i as in Figure 3.6. We may assume $\widetilde{\phi}_V(x_1) = I$, so $\mathscr{D}(v_1) = h(\psi(x_1)) = h\binom{1}{0} = \infty$. From the equation (3.2.9), we have

$$\widetilde{\phi}_V(x_2) = \widetilde{\phi}_V(x_1)\begin{pmatrix} \sigma(s_{23}) & c(s_{23}) \\ 0 & \sigma(s_{23})^{-1} \end{pmatrix}^{-1}\begin{pmatrix} 0 & -c(l_2)^{-1} \\ c(l_2) & 0 \end{pmatrix}^{-1} = \begin{pmatrix} c(s_{23})c(l_2) & * \\ -\sigma(s_{23})c(l_2) & * \end{pmatrix}$$

and $\mathscr{D}(v_2) = \frac{c(s_{23})}{-\sigma(s_{23})}$. Similarly, we obtain $\mathscr{D}(v_3) = 0$ and $\mathscr{D}(v_4) = \sigma(s_{34})c(s_{34})$. Then the cross-ratio $r(\Delta_j, l_3)$ is given by

$$\big[\mathscr{D}(v_3) : \mathscr{D}(v_1) : \mathscr{D}(v_4) : \mathscr{D}(v_2)\big] = \frac{c(s_{23})}{-\sigma(s_{23})\sigma(s_{34})c(s_{34})} = -\frac{c(s_{23})}{\sigma(s_{24})c(s_{34})}.$$

Recall that the cross-ratio $[A : B : C : D] = \frac{(A-D)(B-C)}{(A-C)(B-D)}$. The equation (3.2.12) is obtained from the above equation by replacing $c(s_{23})$ and $c(s_{34})$ through Proposition 3.2.1. $\square$

Remark 3.2.3. These cross-ratios automatically satisfy the gluing equations for $\mathcal{T}$. Namely, the product of the cross-ratios around each edge of $\mathcal{T}$ is equal to 1. Furthermore, they are invariant under the isomorphism Φ.

3.3 Flattenings

Let $\sigma \in Z^1(\partial N; \mathbb{C}^\times)$ and $c \in P_\sigma(\mathcal{T})$. In order to consider log-parameters, we consider an edge of $\mathcal{T}$ without its orientation. However, the vertex-ordering endows each unoriented edge l with an orientation, so $c(l)$ is well-defined without sign-ambiguity.

Recall Proposition 3.2.7 that if Δ_j is positively oriented,

$$\begin{cases} z_j(c) &= \pm\dfrac{\sigma(s_{12})\sigma(s_{45})}{\sigma(s_{24})\sigma(s_{51})}\dfrac{c(l_1)c(l_4)}{c(l_2)c(l_5)} \\ z'_j(c) &= \pm\dfrac{\sigma(s_{53})\sigma(s_{26})}{\sigma(s_{32})\sigma(s_{65})}\dfrac{c(l_2)c(l_5)}{c(l_3)c(l_6)} \\ z''_j(c) &= \pm\dfrac{\sigma(s_{64})\sigma(s_{31})}{\sigma(s_{43})\sigma(s_{16})}\dfrac{c(l_3)c(l_6)}{c(l_1)c(l_4)} \end{cases} \tag{3.3.13}$$

and if Δ_j is negatively oriented,

$$\begin{cases} z_j(c) &= \pm\dfrac{\sigma(s_{24})\sigma(s_{51})}{\sigma(s_{12})\sigma(s_{45})}\dfrac{c(l_2)c(l_5)}{c(l_1)c(l_4)} \\ z'_j(c) &= \pm\dfrac{\sigma(s_{43})\sigma(s_{16})}{\sigma(s_{64})\sigma(s_{31})}\dfrac{c(l_1)c(l_4)}{c(l_3)c(l_6)} \\ z''_j(c) &= \pm\dfrac{\sigma(s_{32})\sigma(s_{65})}{\sigma(s_{53})\sigma(s_{26})}\dfrac{c(l_3)c(l_6)}{c(l_2)c(l_5)} \end{cases} \tag{3.3.14}$$

where $l_1, \cdots, l_6$ are now regarded as unoriented edges. Zickert showed that taking a "logarithm" of the above equations gives a nice flattening. However, we can not directly apply it to our case, since it won't give a flattening. Remark that $\log \circ \sigma : \partial N^1 \to \mathbb{C}$ may not be a cocycle (cf. Equations (3.3.15) and (3.3.16)). We therefore consider the followings sets:

$$\mathbb{A} = \left\{ a \in Z^1(\partial N; \mathbb{C}) \,|\, a \equiv \log \circ \sigma \pmod{\pi i} \right\}$$

$$\mathbb{B} = \left\{ b = (b_1, \cdots, b_h) \,\middle|\, \begin{array}{l} b_j : \pi_1(\Sigma_j) \to \mathbb{C} \text{ homomorphism} \\ \text{such that } b_j \equiv \log \circ \sigma_{\Sigma_j} \pmod{\pi i} \end{array} \right\}$$

It is clear that $(a_{\Sigma_1}, \cdots, a_{\Sigma_h}) \in \mathbb{B}$ for all $a \in \mathbb{A}$. Recall that $a_{\Sigma_j} : \pi_1(\Sigma_j) \to \mathbb{C}$ denotes the homomorphism induced from the cocycle $a \in \mathbb{A}$. The set $\mathbb{B}$ can be identified with $\mathbb{Z}^{2h}$, where $(u_1, v_1, \cdots, u_h, v_h) \in \mathbb{Z}^{2h}$ corresponds to $b = (b_1, \cdots, b_h) \in$

$\mathbb{B}$ given by

$$b_j(\mu_j) = \log \sigma_{\Sigma_j}(\mu_j) + u_j \pi i \text{ and } b_j(\lambda_j) = \log \sigma_{\Sigma_j}(\lambda_j) + v_j \pi i$$

for all $1 \leqslant j \leqslant h$. Recall that $\pi_1(\Sigma_j)$ is an abelian group generated by μ_j and λ_j.

Proposition 3.3.1. The map $\iota : \mathbb{A} \to \mathbb{B},\ a \mapsto (a_{\Sigma_1}, \cdots, a_{\Sigma_h})$ is surjective. In particular, $\mathbb{A}$ is non-empty.

Proof. Let $b = (b_1, \cdots, b_h) \in \mathbb{B}$. We define $a : \partial N^1 \to \mathbb{C}$ on each component Σ_j of ∂N as follows. We choose a spanning tree T on Σ_j. For each unoriented edge e of T we choose any orientation of e and define $a(e) := \log \sigma(e)$ and $a(-e) := -\log \sigma(e)$. For an oriented edge e_0 of Σ_j not in T let $e_1, \cdots, e_m$ be oriented edges of T such that together with e_0 they form a unique cycle γ in $T \cup \{e_0\}$. We define

$$a(e_0) := b_j(\gamma) - a(e_1) - \cdots - a(e_m).$$

Note that $a(e_0) \equiv \log \sigma_{\Sigma_j}(\gamma) - \log \sigma(e_1) - \cdots \log \sigma(e_m) \equiv \log \sigma(e_0)$ in modulo πi. One can check that a is a cocycle satisfying $\iota(a) = b \in \mathbb{B}$ from the fact that the cycle γ forms a fundamental cycle basis. $\square$

We define a flattening $\alpha_j(c, a)$ of each ideal tetrahedron Δ_j of $\mathcal{T}$ (depending on the choice of $c \in P_\sigma(\mathcal{T})$ and $a \in \mathbb{A}$) by defining log parameters α_j^0, α_j^1, and

α_j^2: if Δ_j is positively oriented,

$$\begin{cases} \alpha_j^0 &= \log c(l_1) + \log c(l_4) - \log c(l_2) - \log c(l_5) \\ & \quad +a(s_{12}) + a(s_{45}) - a(s_{24}) - a(s_{51}), \\ \alpha_j^1 &= \log c(l_2) + \log c(l_5) - \log c(l_3) - \log c(l_6) \\ & \quad +a(s_{53}) + a(s_{26}) - a(s_{32}) - a(s_{65}), \\ \alpha_j^2 &= \log c(l_3) + \log c(l_6) - \log c(l_1) - \log c(l_4) \\ & \quad +a(s_{64}) + a(s_{31}) - a(s_{43}) - a(s_{16}) \end{cases} \tag{3.3.15}$$

and if Δ_j is negatively oriented,

$$\begin{cases} \alpha_j^0 &= \log c(l_2) + \log c(l_5) - \log c(l_1) - \log c(l_4) \\ & \quad +a(s_{24}) + a(s_{51}) - a(s_{12}) - a(s_{45}), \\ \alpha_j^1 &= \log c(l_1) + \log c(l_4) - \log c(l_3) - \log c(l_6) \\ & \quad +a(s_{43}) + a(s_{16}) - a(s_{64}) - a(s_{31}) \\ \alpha_j^2 &= \log c(l_3) + \log c(l_6) - \log c(l_2) - \log c(l_5) \\ & \quad +a(s_{32}) + a(s_{65}) - a(s_{53}) - a(s_{26}) \end{cases} \tag{3.3.16}$$

for Figure 3.6. Note that $\alpha_j(c,a)$ is indeed a flattening of Δ_j. Namely, $\alpha_j^0 + \alpha_j^1 + \alpha_j^2 = 0$ and $\alpha_j^0 \equiv \log z_j$, $\alpha_j^1 \equiv \log z_j'$, $\alpha_j^2 \equiv \log z_j''$ in modulo πi, since $a \in \mathbb{A}$ is a cocycle that agrees with $\log \circ \sigma$ in modulo πi.

Following Theorem 3.1.2 (cf. the equation (3.1.1)), we define the map

$$\Psi : P_\sigma(\mathcal{T}) \times \mathbb{A} \to \mathbb{C}/\pi^2\mathbb{Z},\ (c,a) \mapsto \sum_{j=1}^{n} \epsilon_j \, R(\alpha_j(c,a)).$$

Proposition 3.3.2. $\Psi(c,a) = \Psi(c,a')$ if $\iota(a) = \iota(a') \in \mathbb{B}$.

Proof. Since a and a' induce the same element of $\mathbb{B}$, there exists $\theta \in C^0(\partial N; \mathbb{C})$ satisfying $a' = a^\theta$. As in the proof of Proposition 3.2.3, we may assume that θ

is trivial except on a single vertex x_0 and $\theta(x_0) = \pi i$. Let l_0 be the long-edge of N having x_0 as an endpoint, and $\Delta_1, \cdots, \Delta_m$ be the tetrahedra of $\mathcal{T}$ containing l_0. Let $\alpha_j(c,a) = (z_j; p_j, q_j)$ and $\alpha_j(c,a') = (z_j; p'_j, q'_j)$ be the flattenings of Δ_j given by the equation (3.3.15) or (3.3.16), where z_j is the shape parameter of Δ_j at l_0. One can check that $p'_j = p_j$ and $q'_j = q_j + 1$ for all $1 \leqslant j \leqslant m$. Therefore, we have

$$\Psi(c,a') - \Psi(c,a) = \frac{\pi i}{2}\sum_{j=1}^{m}(\epsilon_j \log z_j) \equiv \frac{\pi i}{2}\log\prod_{j=1}^{m} z_j^{\epsilon_j} \equiv 0 \mod \pi^2\mathbb{Z}.$$

For the last equality we use Remark 3.2.3. □

We therefore obtain the induced map, also denoted by Ψ,

$$\Psi : P_\sigma(\mathcal{T}) \times \mathbb{B} \to \mathbb{C}/\pi^2\mathbb{Z}$$

by defining $\Psi(c,b) := \Psi(c,a)$ for any $a \in \mathbb{A}$ such that $\iota(a) = b \in \mathbb{B}$.

3.3.1 Main theorem

Recall that for $\kappa = (r_1, s_1, \cdots, r_h, s_h)$ the manifold N_κ is obtained from N by performing a Dehn filling that kills the curve $r_j\mu_j + s_j\lambda_j$ on each Σ_j, where (r_j, s_j) is either a pair of coprime integers or the symbol ∞ meaning that we do not fill Σ_j.

Let $c \in P_\sigma(\mathcal{T})$ such that the representation $\rho_c : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ factors through N_κ as a $\mathrm{PSL}(2,\mathbb{C})$-representation. If N_κ has a boundary, i.e. $(r_i, s_i) = \infty$ for some i, then we further assume that the induced representation ρ_c is a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation, so that the complex volume of ρ_c are well-defined. This exactly happens when

$$\begin{cases} \mathrm{tr}(\rho_c(\mu_j)) = \pm 2,\ \mathrm{tr}(\rho_c(\lambda_j)) = \pm 2 & \text{if } (r_j, s_j) = \infty \\ \rho_c(\mu_j^{r_j}\lambda_j^{s_j}) = \pm I & \text{if } (r_j, s_j) \neq \infty \end{cases}$$

and in this case, the equation (3.2.6) tells us that

$$\begin{cases} \sigma_{\Sigma_j}(\mu_j) = \pm 1, \ \sigma_{\Sigma_j}(\lambda_j) = \pm 1 & \text{for all } (r_j, s_j) = \infty \\ \sigma_{\Sigma_j}(\mu_j^{r_j} \lambda_j^{s_j}) = \pm 1 & \text{for all } (r_j, s_j) \neq \infty. \end{cases}$$

Therefore there exists an element $b = (b_1, \cdots, b_h) \in \mathbb{B}$ satisfying

$$\begin{cases} b_j(\mu_j) = b_j(\lambda_j) = 0 & \text{for all } (r_j, s_j) = \infty \\ b_j(\mu_j^{r_j} \lambda_j^{s_j}) = 0 & \text{for all } (r_j, s_j) \neq \infty. \end{cases} \tag{3.3.17}$$

Theorem 3.3.1. Suppose that the representation $\rho_c : \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$ factors $\pi_1(N_\kappa)$ as a $\mathrm{PSL}(2, \mathbb{C})$-representation and induces a $(\mathrm{PSL}(2, \mathbb{C}), P)$-representation $\rho_c : \pi_1(N_\kappa) \to \mathrm{PSL}(2, \mathbb{C})$. Then the complex volume of ρ_c is given by

$$i\mathrm{Vol}_{\mathbb{C}}(\rho_c) \equiv \Psi(c, b) \mod \frac{1}{2}\pi^2\mathbb{Z} \tag{3.3.18}$$

for $b = (b_1, \cdots, b_h) \in \mathbb{B}$ satisfying the equation (3.3.17).

Proof. Let $a \in \mathbb{A}$ satisfying $\iota(a) = b$ and let $\alpha_j(c, a)$ be the flattening of Δ_j given by the equation (3.3.15) or (3.3.16). Let us rewrite the equations (3.3.15) and (3.3.16) as follows (note that $a \in \mathbb{A}$ is a cocycle) : if Δ_j is positively oriented,

$$\begin{cases} \alpha_j^0 & = & \log c(l_1) - \log c(l_2) - a(s_{31}) + a(s_{12}) - a(s_{23}) \\ & & \quad + \log c(l_4) - \log c(l_5) - a(s_{34}) + a(s_{45}) - a(s_{53}), \\ \alpha_j^1 & = & \log c(l_5) - \log c(l_3) - a(s_{45}) + a(s_{53}) - a(s_{34}) \\ & & \quad + \log c(l_2) - \log c(l_6) - a(s_{42}) + a(s_{26}) - a(s_{64}), \\ \alpha_j^2 & = & \log c(l_6) - \log c(l_4) - a(s_{26}) + a(s_{64}) - a(s_{42}) \\ & & \quad + \log c(l_3) - \log c(l_1) - a(s_{23}) + a(s_{31}) - a(s_{12}) \end{cases} \tag{3.3.19}$$

and if Δ_j is negatively oriented,

$$\begin{cases} \alpha_j^0 &= -\log c(l_1) + \log c(l_2) + a(s_{31}) - a(s_{12}) + a(s_{23}) \\ & \quad -\log c(l_4) + \log c(l_5) + a(s_{34}) - a(s_{45}) + a(s_{53}), \\ \alpha_j^1 &= -\log c(l_6) + \log c(l_4) + a(s_{26}) - a(s_{64}) + a(s_{42}) \\ & \quad -\log c(l_3) + \log c(l_1) + a(s_{23}) - a(s_{31}) + a(s_{12}), \\ \alpha_j^2 &= -\log c(l_5) + \log c(l_3) + a(s_{45}) - a(s_{53}) + a(s_{34}) \\ & \quad -\log c(l_2) + \log c(l_6) + a(s_{42}) - a(s_{26}) + a(s_{64}) \end{cases} \tag{3.3.20}$$

for Figure 3.6. Note that each log-parameter in the equations (3.3.19) and (3.3.20) consists of ten terms, where the first five terms lie on a single face of Δ_j and the other five terms also lie on another face of Δ_j.

Claim 1. The sum of log-parameters around each edge of $\mathcal{T}$ is zero.

Proof of Claim 1. Let us consider the log-parameters around an edge l_0 of $\mathcal{T}$. We denote edges around l_0 by $l_1, l_2, \cdots, l_{2m-1}, l_{2m}$ as in Figure 3.7 and denote the short-edge joining from l_i to l_j by s_{ij}.

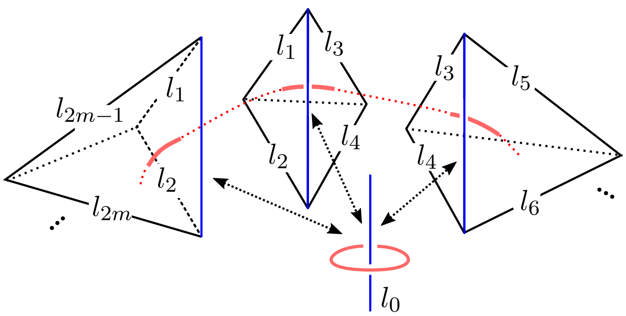


Figure 3.7: Log-parameters around an edge l_0

Then the sum of log-parameters around l_0 is given by

$$-\log c(l_1) + \log c(l_2) - a(s_{02}) + a(s_{21}) - a(s_{10})$$

$$+\log c(l_3) - \log c(l_4) - a(s_{03}) + a(s_{34}) - a(s_{40})$$
$$-\log c(l_3) + \log c(l_4) - a(s_{04}) + a(s_{43}) - a(s_{30})$$
$$+\log c(l_5) - \log c(l_6) - a(s_{05}) + a(s_{56}) - a(s_{60})$$
$$\cdots$$
$$-\log c(l_{2m-1}) + \log c(l_{2m}) - a(s_{0(2m)}) + a(s_{(2m)(2m-1)}) - a(s_{(2m-1)0})$$
$$+\log c(l_1) - \log c(l_2) - a(s_{01}) + a(s_{12}) - a(s_{20})$$

and is canceled out to zero, since $a(s_{ij}) = -a(s_{ji})$. □

Claim 2. The sum of log-parameters along a normal path γ in the neighborhood of an ideal vertex v_j of $\mathcal{T}$, corresponding to Σ_j, is $2b_j(\gamma)$.

Proof of Claim 2. The proof of [Zic09, Theorem 6.5] exactly tells us that the sum of $\log c$-terms along γ is canceled out to zero. Therefore we may consider the sum of a-terms only.

As γ crosses a face, it picks up three a-terms as it enters to the face and also picks up another three a-terms as it departs the face. More precisely, suppose γ crosses a face whose edge are denoted by l_1, l_2, and l_3 as in Figure 3.8. As γ

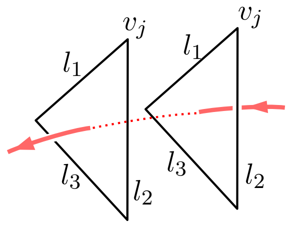


Figure 3.8: A normal path crossing a face

enters to the face, it may pass either l_1 or l_2. From the equations (3.3.19) and (3.3.20), one can check that it picks up $a(s_{31}) + a(s_{32}) + a(s_{12})$ if γ passes l_1; $a(s_{31}) + a(s_{32}) + a(s_{21})$ if γ passes l_2. Similarly, as γ departs the face, it picks up $a(s_{13}) + a(s_{23}) + a(s_{21})$ if γ passes l_1; $a(s_{13}) + a(s_{23}) + a(s_{12})$ if γ passes l_2.

Summing up the cases, we have $2a(s_{12})$ if γ passes l_1 and l_2 in order; $2a(s_{21})$ if γ passes l_2 and l_1 in order; zero, otherwise. Therefore, the sum of a-terms along γ results in $2b_j(\gamma)$. See also Figure 3.9. Recall that $b_j : \pi_1(\Sigma_j) \to \mathbb{C}$ is the induced homomorphism from $a \in \mathbb{A}$. $\square$

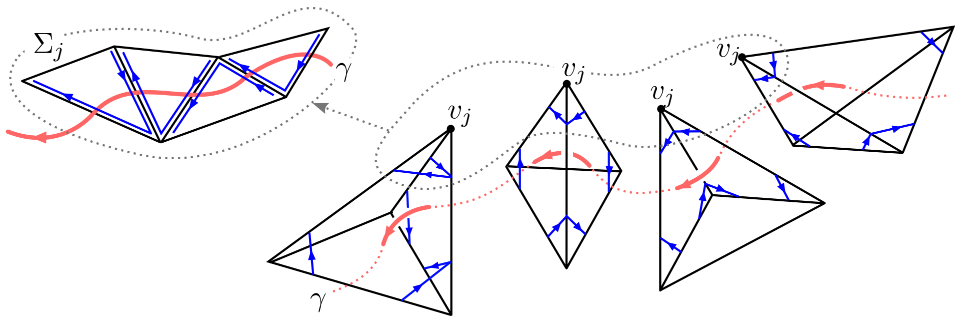


Figure 3.9: Log-parameters along a normal path γ

Claims 1 and 2 tell us that if we choose $b \in \mathbb{B}$ as in the equation (3.3.17), then the flattenings $\alpha_j(c, a)$ satisfy the edge, cusp, filling conditions in Theorem 3.1.2. Finally, the theorem follows form [Neu04, Lemma 11.3], which says that if the flattenings $\alpha_j(c, a)$ satisfy the conditions of Theorem 3.1.2 except the parity condition, then the equation (3.3.18) holds in modulo $\frac{1}{2}\pi^2\mathbb{Z}$. $\square$

Remark 3.3.1. As in [Neu04] or [Zic09, Remark 6.7], parity along normal curves can be viewed as an element of $\mathrm{Ker}(H^1(N;\mathbb{Z}/2) \to H^1(\partial N;\mathbb{Z}/2))$. Therefore, if N is a link exterior in the 3-sphere, then we have the trivial kernel and Theorem 3.3.1 holds also in modulo $\pi^2\mathbb{Z}$.

Example 3.3.1. Let us continue Example 3.2.1 of the figure-eight knot complement. Assigning vertex-orderings of Δ_1 and Δ_2 as in Figure 3.4, we have $\epsilon_1 = 1$ and $\epsilon_2 = -1$. To consider $\kappa = (r, s)$-Dehn filling on the knot complement, we need a pair (M, L) satisfying $M^r L^s = 1$ and the equation (3.2.7), the A-polynomial of the knot. Among all the possibilities, we choose one that maximizes the volume in order to find the geometric one (see [Thu78, Fra04]). Using Mathematica, for instance, we choose (M, L) as follows.

κ	$(M,\, L)$	(u,v)
$(1,5)$	$(0.840595+0.007451i,\ -0.838678-0.607067i)$	$(4,0)$
$(2,5)$	$(0.841492+0.014849i,\ -0.871207-0.623622i)$	$(2,0)$
$(3,5)$	$(0.842985+0.022140i,\ -0.906286-0.636885i)$	$(-2,2)$
$(4,5)$	$(0.845070+0.029264i,\ -0.721385-0.494189i)$	$(1,0)$

For each given pair (M, L) one can check that $P_\sigma(\mathcal{T})$ consists of a single element, say $c : \{l_1, l_2\} \to \mathbb{C}$ with $c(l_2) = 1$, up to the diagonal action.

We then need $b \in \mathbb{B}$ satisfying $b(\mu^r\lambda^s) = 0$, or equivalently $(u, v) \in \mathbb{Z}^2$ satisfying

$$r\,(\log M + u\pi i) + s\,(\log L + v\pi i) = 0.$$

Recall that $b(\mu) = \log M + u\pi i$ and $b(\lambda) = \log L + v\pi i$. One can check that such (u, v) is given as in the above table. We also choose $a \in \mathbb{A}$ satisfying $\iota(a) = b$ as follows: $a(s_4) = a(s_7) = a(s_{10}) = 0$, $a(s_2) = a(s_5) = a(s_8) = a(s_{11}) = b(\mu)$, $a(s_6) = a(s_9) = a(s_{12}) = -b(\mu)$, $a(s_3) = -b(\lambda) + b(\mu)$, and $a(s_1) = b(\lambda) - 2b(\mu)$. (Compare the definition of a with that of σ in Example 3.2.1.)

Let z_1 be the cross-ratio parameter of Δ_1 at the edge $\overline{12}$ and z_2 be the cross-ratio. parameter of Δ_2 at the edge $\overline{03}$. From Proposition 3.2.12 and the equations (3.3.15) and (3.3.16), the flattening $\alpha_1(c, a) = (z_1; p_1, q_1)$ of Δ_1 is given by

$$\begin{cases} z_1 &= \frac{LM^4 c(l_1)^2}{c(l_2)^2} \\ p_1 &= \frac{1}{\pi i}\left[b(\lambda) + 4b(\mu) + 2\log c(l_1) - 2\log c(l_2) - \log z_1\right] \\ q_1 &= \frac{1}{\pi i}\left[-b(\lambda) - 2b(\mu) - \log c(l_1) + \log c(l_2) + \log(1 - z_1)\right] \end{cases}$$

and the flattening $\alpha_2(c,a) = (z_2; p_2, q_2)$ of Δ_2 is given by

$$\begin{cases} z_2 &= \frac{1}{L}\frac{c(l_2)^2}{c(l_1)^2} \\ p_2 &= \frac{1}{\pi i}\left[-b(\lambda) + 2\log c(l_2) - 2\log c(l_1) - \log z_2\right] \\ q_2 &= \frac{1}{\pi i}\left[b(\lambda) + \log c(l_1) - \log c(l_2) + \log(1 - z_2)\right]. \end{cases}$$

Finally, i times the complex volumes are given by $\Psi(c,b) = R(z_1; p_1, q_1) - R(z_2; p_2, q_2)$ as follows. These complex volumes coincide with the one given by Snappy in modulo $\pi^2\mathbb{Z}$ (see Remark 3.3.1).

κ	$\Psi(c,b)$
$(1,5)$	$1.967879974 + 1.918602377i$
$(2,5)$	$5.909776683 + 1.919520361i$
$(3,5)$	$3.930060763 + 1.921026911i$
$(4,5)$	$7.872366052 + 1.923087332i$

Chapter 4

Potential functions

For a diagram of a link L in S^3, Cho and Murakami [CM13] (motivated from the work of Yokota [Yok02]) defined the potential function whose critical point, slightly different from the usual sense, corresponds to a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation of $\pi_1(S^3\backslash L)$. They proved that the complex volume of such representations can be computed from the potential function with its partial derivatives. In this chapter, we extend the potential function to an arbitrary $\mathrm{PSL}(2,\mathbb{C})$-representation and, under a mild assumption, we present a combinatorial formula for computing the complex volume of a $\mathrm{PSL}(2,\mathbb{C})$-representation of a closed 3-manifold.

4.1 Generalized potential functions

Let L be a link in S^3 with h components. Throughout the chapter, we fix an oriented diagram, denoted also by L, of L. We assume that every component of L has at least one over-passing and under-passing crossing, respectively, so that we can consider the octahedral decomposition $\mathcal{O}$ of $S^3\backslash(L \cup \{p, q\})$ where $p, q \in S^3$ are two points not in L. Such a decomposition was introduced in [Thu99] and can be found in several articles, such as [Yok02, Wee05, Cho16a, KKY16]. See also Section 5.1.2.

We denote the number of the regions of L by n and assign a complex variable

w_j $(1 \leqslant j \leqslant n)$ to each region of L. We let $\mathbf{w} = (w_1, \cdots, w_n)$. We also assign a complex variable m_i $(1 \leqslant i \leqslant h)$ to each component of L and let $\mathbf{m} = (m_1, \cdots, m_h)$. For notational simplicity, we enumerate a region and a component of L by the index of the variables assigned to them. For each crossing, say c, of L we define

$$\begin{aligned}\mathbb{W}_c(\mathbf{w},\mathbf{m}) := \mathrm{Li}_2\left(\frac{w_m}{m_\beta w_j}\right) + \mathrm{Li}_2\left(\frac{w_k}{m_\alpha w_j}\right) - \mathrm{Li}_2\left(\frac{w_l}{m_\beta w_k}\right) - \mathrm{Li}_2\left(\frac{w_l}{m_\alpha w_m}\right)\\ + \mathrm{Li}_2\left(\frac{w_j w_l}{w_m w_k}\right) - \frac{\pi^2}{6} + \log\left(\frac{w_m}{m_\beta w_j}\right)\log\left(\frac{w_k}{m_\alpha w_j}\right)\end{aligned}$$

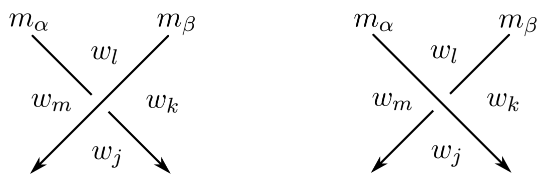
for Figure 4.1(a) and

$$\begin{aligned}\mathbb{W}_c(\mathbf{w},\mathbf{m}) := -\mathrm{Li}_2\left(\frac{m_\beta w_m}{w_j}\right) - \mathrm{Li}_2\left(\frac{m_\alpha w_k}{w_j}\right) + \mathrm{Li}_2\left(\frac{m_\beta w_l}{w_k}\right) + \mathrm{Li}_2\left(\frac{m_\alpha w_l}{w_m}\right)\\ - \mathrm{Li}_2\left(\frac{w_j w_l}{w_m w_k}\right) + \frac{\pi^2}{6} - \log\left(\frac{m_\beta w_m}{w_j}\right)\log\left(\frac{m_\alpha w_k}{w_j}\right)\end{aligned}$$

for Figure 4.1(b). We remark that each dilogarithm term of $\mathbb{W}_c$ corresponds to an ideal triangulation (see Figure 4.2 or 4.3). We then define the *generalized potential function*

$$\mathbb{W}(\mathbf{w},\mathbf{m}) := \sum_{\text{crossing } c} \mathbb{W}_c(\mathbf{w},\mathbf{m})$$

where the sum is over all crossings of L.



(a) Positive crossing (b) Negative crossing

Figure 4.1: Variables around a crossing

Remark 4.1.1. The generalized potential function $\mathbb{W}$ reduces to the potential function W in [CM13] or [Cho16b] when $m_1 = \cdots = m_h = 1$.

Definition 4.1.1. (i) A point $(\mathbf{w}, \mathbf{m}) \in (\mathbb{C}\backslash\{0\})^{n+h} = (\mathbb{C}^\times)^{n+h}$ is called a *solution* if

$$\exp\left(w_j \frac{\partial \mathbb{W}}{\partial w_j}\right) = 1 \quad \text{for all } 1 \leqslant j \leqslant n. \tag{4.1.1}$$

(ii) A point $(\mathbf{w}, \mathbf{m})$ is said to be *non-degenerate* if the following five values are not 1 at each crossing of L:

$$\begin{cases} \dfrac{w_m}{m_\beta w_j}, \dfrac{w_k}{m_\alpha w_j}, \dfrac{w_l}{m_\beta w_k}, \dfrac{w_l}{m_\alpha w_m}, \dfrac{w_j w_l}{w_m w_k} & \text{for Figure 4.1(a)} \\ \dfrac{m_\beta w_m}{w_j}, \dfrac{m_\alpha w_k}{w_j}, \dfrac{m_\beta w_l}{w_k}, \dfrac{m_\alpha w_l}{w_m}, \dfrac{w_j w_l}{w_k w_m} & \text{for Figure 4.1(b).} \end{cases} \tag{4.1.2}$$

Theorem 4.1.1 (Theorem 1.2.1)**.** A non-degenerate solution $(\mathbf{w}, \mathbf{m})$ corresponds to a representation $\rho_{\mathbf{w},\mathbf{m}} : \pi_1(S^3\backslash L) \to \mathrm{PSL}(2, \mathbb{C})$ such that the eigenvalues of $\rho_{\mathbf{w},\mathbf{m}}(\mu_i)$ are m_i and m_i^{-1} up to sign for all $1 \leqslant i \leqslant h$. Here μ_i denotes a meridian of the i-th component of L.

4.1.1 Proof of Theorem 4.1.1

Following [Cho16a], we subdivide each ideal octahedron of $\mathcal{O}$ into five ideal tetrahedra as in Figures 4.2 and 4.3. We denote by $\mathcal{T}$ the resulting ideal triangulation of $S^3\backslash(L \cup \{p, q\})$. For a given non-degenerate solution $(\mathbf{w}, \mathbf{m})$ we assign the cross-ratio to each ideal tetrahedron of $\mathcal{T}$ as in Figures 4.2 and 4.3. The equation (4.1.2) guarantees that these tetrahedra are non-degenerated. The product of the cross-ratios around each of edges that are created to divide the octahedra into tetrahedra is 1:

$$\begin{cases} \dfrac{w_m}{m_\beta w_j} \dfrac{m_\beta w_k}{w_l} \dfrac{w_j w_l}{w_m w_k} = 1 = \dfrac{w_k}{m_\alpha w_j} \dfrac{m_\alpha w_m}{w_l} \dfrac{w_j w_l}{w_m w_k} & \text{for Figure 4.1(a)} \\ \dfrac{m_\alpha w_l}{w_m} \dfrac{w_j}{m_\alpha w_k} \dfrac{w_k w_m}{w_j w_l} = 1 = \dfrac{w_j}{m_\beta w_m} \dfrac{m_\beta w_l}{w_k} \dfrac{w_k w_m}{w_j w_l} & \text{for Figure 4.1(b).} \end{cases}$$

Therefore, at each crossing, five tetrahedra are well-glued to form an octahedron.

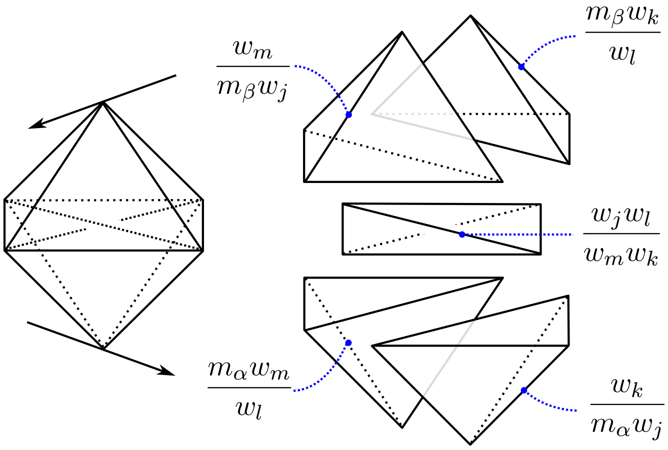


Figure 4.2: Cross-ratios for Figure 4.1(a)

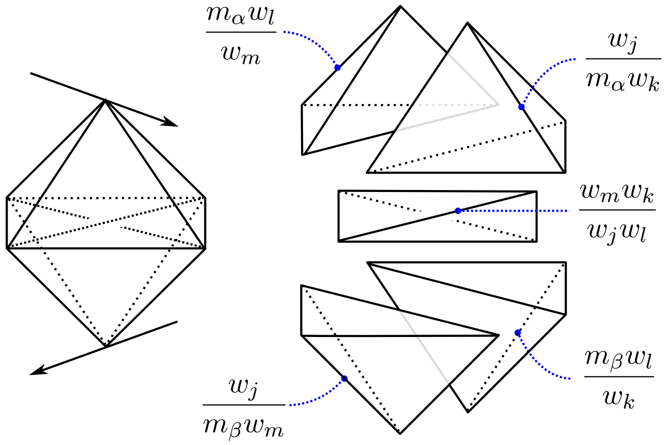


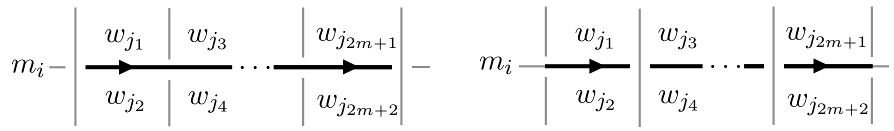
Figure 4.3: Cross-ratios for Figure 4.1(b)

We now check that the given cross-ratios satisfy the gluing equations for $\mathcal{O}$, i.e. the product of the cross-ratios around each edge of $\mathcal{O}$ is 1. We thus shall obtain a representation

$$\rho_{\mathbf{w},\mathbf{m}} : \pi_1(S^3 \backslash (L \cup \{p, q\})) = \pi_1(S^3 \backslash L) \to \mathrm{PSL}(2, \mathbb{C})$$

up to conjugation as a holonomy representation. We note that a similar computation can be found in [Cho16a] and [KKY16]

Recall that L has n regions, so $n-2$ crossings, $n-2$ over-arcs and $n-2$ under-arcs. Here an over (resp., under)-arc is a maximal part of L that does not under (resp., over)-pass a crossing. See Figure 4.4. Recall also that the octahedral decomposition $\mathcal{O}$ has $3n-4$ edges; (i) n *regional edges* corresponding to the regions; (ii) $n-2$ *over-edges* corresponding to the over-arcs; (iii) $n-2$ *under-edges* corresponding to the under-arcs. We refer to [KKY16, §3] for details.



(a) Over-arc (b) Under-arc

Figure 4.4: Over- and under-arcs

Suppose an over-arc of L over-passes m crossings as in Figure 4.4(a). Then around the corresponding over-edge, there are $4m+2$ cross-ratios; each of the over-passed crossings contributes 4 cross-ratios, and two crossings coming from the ends of the over-arc respectively contributes one cross-ratio (cf. Figure 10 in [KKY16]). The product of these cross-ratios is

$$\left(\frac{w_{j_1}}{m_i w_{j_2}}\right) \cdot \left(\frac{m_i w_{j_1}}{w_{j_2}} \frac{w_{j_4}}{m_i w_{j_3}}\right)^{-1} \cdots \left(\frac{m_i w_{j_{2m-1}}}{w_{j_{2m}}} \frac{w_{j_{2m+2}}}{m_i w_{j_{2m+1}}}\right)^{-1} \cdot \left(\frac{m_i w_{j_{2m+2}}}{w_{j_{2m+1}}}\right) = 1$$

for Figure 4.4(a). Similarly, the product of cross-ratios around an under-edge is

1 :

$$\left(\frac{w_{j_2}}{m_i w_{j_1}}\right) \cdot \left(\frac{m_i w_{j_2}}{w_{j_1}} \frac{w_{j_3}}{m_i w_{j_4}}\right)^{-1} \cdots \left(\frac{m_i w_{j_{2m}}}{w_{j_{2m-1}}} \frac{w_{j_{2m+1}}}{m_i w_{j_{2m+2}}}\right)^{-1} \cdot \left(\frac{m_i w_{j_{2m+1}}}{w_{j_{2m+2}}}\right) = 1$$

for Figure 4.4(b).

Suppose a region of L has m crossings (or corners). The corresponding regional edge is represented by a horizontal edge of the octahedron at each of these crossings. Therefore, there are $3m$ cross-ratios around the regional edge. See Figures 4.2 and 4.3 that three cross-ratios are attached to each horizontal edge. Let $\tau_{c,j}$ be the product of cross-ratios coming from a crossing c and attached to the regional edge corresponding to the j-th region. Then it is clear that the product of the cross-ratios around the regional edge corresponding to the j-th region is given by

$$\prod_{\text{crossing } c} \tau_{c,j} \tag{4.1.3}$$

where the product is over all crossings appeared in the j-th region. On the other hand, τ-values can be directly computed as follows from the cross-ratios given in Figures 4.2 and 4.3 :

$$\begin{cases} \tau_{c,l} = \dfrac{(\frac{1}{m_\beta} w_l - w_k)(\frac{1}{m_\alpha} w_l - w_m)}{w_k w_m - w_j w_l}, & \tau_{c,k} = \dfrac{w_j w_l - w_k w_m}{(\frac{1}{m_\alpha} w_k - w_j)(m_\beta w_k - w_l)} \\ \tau_{c,m} = \dfrac{w_j w_l - w_k w_m}{(\frac{1}{m_\beta} w_m - w_j)(m_\alpha w_m - w_l)}, & \tau_{c,j} = \dfrac{(m_\alpha w_j - w_k)(m_\beta w_j - w_m)}{w_k w_m - w_j w_l} \end{cases}$$

for Figure 4.1(a) and

$$\begin{cases} \tau_{c,l} = \dfrac{w_k w_m - w_j w_l}{(m_\beta w_l - w_k)(m_\alpha w_l - w_m)}, & \tau_{c,k} = \dfrac{(m_\alpha w_k - w_j)(\frac{1}{m_\beta} w_k - w_l)}{w_j w_l - w_k w_m} \\ \tau_{c,m} = \dfrac{(m_\beta w_m - w_j)(\frac{1}{m_\alpha} w_m - w_l)}{w_j w_l - w_k w_m}, & \tau_{c,j} = \dfrac{w_k w_m - w_j w_l}{(\frac{1}{m_\alpha} w_j - w_k)(\frac{1}{m_\beta} w_j - w_m)} \end{cases}$$

for Figure 4.1(b). Furthermore, a straightforward computation shows that

$$\tau_{c,j} = \exp\left(w_j \frac{\partial \mathbb{W}_c}{\partial w_j}\right)$$

holds for any crossing c and any region. It thus follows from the equation (1.2.3) that the τ-product in the equation (4.1.3) is 1. Namely, the product of the cross-ratios around each regional edges is 1.

Remark 4.1.2. Rewriting the equation (4.1.1) as the equation (4.1.3), one can checked that the equation (4.1.1) is invariant under change $m_i \mapsto \frac{1}{m_i}$ for all $1 \leqslant i \leqslant h$.

We finally claim that the eigenvalues of $\rho_{\mathbf{w},\mathbf{m}}(\mu_i)$ are m_i and m_i^{-1} (up to sign). Since we assume that each component of L has at least one over-passing crossing and at least one under-passing crossing, it contains a local diagram as in Figure 4.5 (left). Then a meridian μ_i (up to base point) passes through two ideal tetrahedra coming from the ends as in Figure 4.5 (middle). Therefore, the scaling factor of the holonomy action for μ_i is given by the product of two cross-ratios

$$\left(\frac{w_j}{m_i w_k}\right)^{-1} \frac{m_i w_j}{w_k} = m_i^2.$$

It follows that the eigenvalues of $\rho_{\mathbf{w},\mathbf{m}}(\mu_i) \in \mathrm{PSL}(2,\mathbb{C})$ are m_i and m_i^{-1} up to sign.

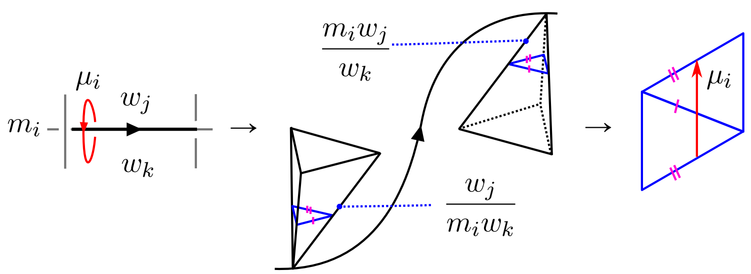


Figure 4.5: A meridian

4.2 Relation with a Ptolemy assignment

Let us briefly recall the notion of a deformed Ptolemy assignment (Section 3.2) which is the key ingredient for proving Theorems 1.2.2 and 1.2.3.

Replacing each ideal tetrahedron of $\mathcal{T}$ by a truncated tetrahedron, we obtain a compact 3-manifold, say N, whose interior is homeomorphic to $S^3 \backslash (L \cup \{p, q\})$. Recall that a truncated tetrahedron is a polyhedron obtained from a tetrahedron by chopping off a small neighborhood of each vertex; see Figure 3.3. Note that the boundary ∂N is triangulated and is consisted of h tori with two spheres. We denote by N^i and ∂N^i the set of the oriented i-cells (unoriented when $i = 0$) of N and ∂N, respectively. We call an 1-cell of ∂N a *short edge* and call an 1-cell of N not in ∂N a *long edge*. We denote by $\mathcal{T}^1$ the set of the oriented 1-cells of $\mathcal{T}$ and identify each edge of $\mathcal{T}$ with a long-edge of N in a natural way.

An assignment $\sigma : \partial N^1 \to \mathbb{C}^\times$ is called a *cocycle* if (i) $\sigma(e)\sigma(-e) = 1$ for all $e \in \partial N^1$; (ii) $\sigma(e_1)\sigma(e_2)\sigma(e_3) = 1$ whenever e_1, e_2, and e_3 bound, respecting an orientation, a 2-cell in ∂N. A cocycle $\sigma : \partial N^1 \to \mathbb{C}^\times$ induces a homomorphism $\pi_1(\Sigma) \to \mathbb{C}^\times$ on each component Σ of ∂N. For notational simplicity we denote all of such homomorphisms by $\overline{\sigma}$.

Definition 4.2.1. For a given cocycle $\sigma : \partial N^1 \to \mathbb{C}^\times$, an assignment $c : \mathcal{T}^1 \to \mathbb{C}^\times$ is called a *σ-deformed Ptolemy assignment* if $c(-e) = -c(e)$ for all $e \in \mathcal{T}^1$ and

$$c(l_3)c(l_6) = \frac{\sigma(s_{23})}{\sigma(s_{35})}\frac{\sigma(s_{26})}{\sigma(s_{65})}c(l_2)c(l_5) + \frac{\sigma(s_{13})}{\sigma(s_{34})}\frac{\sigma(s_{16})}{\sigma(s_{64})}c(l_1)c(l_4)$$

for each ideal tetrahedron Δ of $\mathcal{T}$. Here l_i's denote 1-cells of Δ and s_{ij} denotes the 1-cell in $\partial N \cap \Delta$ running from l_i to l_j as in Figure 3.3.

Recall that a σ-deformed Ptolemy assignment c corresponds to an assignment $\phi : N^1 \to \mathrm{SL}(2, \mathbb{C})$ satisfying cocycle condition. It thus corresponds to a representation $\rho_c : \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$ up to conjugation. The cocycle ϕ can be explicitly given as follows:

$$\phi(l_j) = \begin{pmatrix} 0 & -c(l_j)^{-1} \\ c(l_j) & 0 \end{pmatrix}, \quad \phi(s_{ij}) = \begin{pmatrix} \sigma(s_{ij}) & -\frac{\sigma(s_{ki})}{\sigma(s_{jk})}\frac{c(l_k)}{c(l_i)c(l_j)} \\ 0 & \sigma(s_{ij})^{-1} \end{pmatrix}$$

where the index k is chosen so that l_k and s_{ij} lie on the same 2-cell. Also, c determines the cross-ratio of each ideal tetrahedron of $\mathcal{T}$; see Proposition 3.2.7. For instance, the cross-ratio at l_3 in Figure 3.3 is given by

$$\frac{\sigma(s_{12})\sigma(s_{45})}{\sigma(s_{24})\sigma(s_{51})}\frac{c(l_1)c(l_4)}{c(l_2)c(l_5)} \in \mathbb{C}\backslash\{0,1\}.$$

Recall Remark 3.2.3 that these cross-ratios are non-degenerate and satisfy the gluing equations for $\mathcal{T}$ such that the holonomy representation coincides with ρ_c.

The following proposition shows how a σ-deformed Ptolemy assignment is related to the variables $\mathbf{w}$ and $\mathbf{m}$ in Section 4.1. Recall that $\mathcal{T}$ has n regional edges, each of which corresponds to a region of L. We orient these edges so that their initial points are the same (see Figures 5.5 and 4.8), and denote them by e_j $(1 \leqslant j \leqslant n)$ according to the index of regions. Note that these edges appear as horizontal edges of an octahedron as in Figure 4.6 (cf. Figure 4.1).

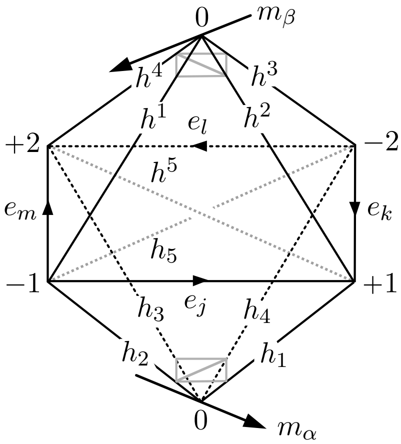
Proposition 4.2.1. Let $\sigma : \partial N^1 \to \mathbb{C}^\times$ be a cocycle trivial on the sphere components. Then for any σ-deformed Ptolemy assignment $c : \mathcal{T}^1 \to \mathbb{C}^\times$,

$$(\mathbf{w}, \mathbf{m}) = \big(c(e_1), \cdots, c(e_n), \overline{\sigma}(\mu_1), \cdots, \overline{\sigma}(\mu_h)\big)$$

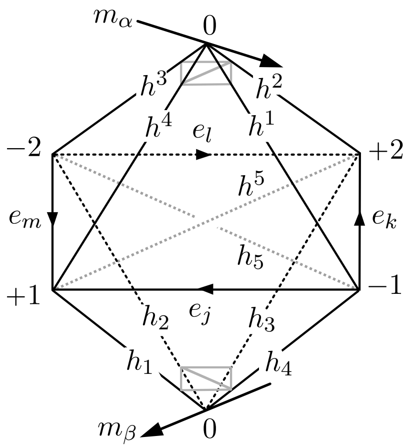
is a non-degenerate solution such that $\rho_{\mathbf{w},\mathbf{m}}$ coincides with ρ_c, viewed as a $\mathrm{PSL}(2,\mathbb{C})$-representation, up to conjugation.

Proof. At each crossing of L, we denote edges of $\mathcal{T}$ as in Figure 4.6. We orient these edges so that they coherent with the vertex-ordering given as in Figure 4.6. Recall that h^2 and h^4 are identified in $\mathcal{T}$ and so are h_2 and h_4. We denote by s^{ij}

(resp., s_{ij}) the short-edge running from h^i to h^j (resp., h_i to h_j). For instance, s^{42} and s_{42} are short-edges winding the over-arc and under-arc, respectively.



(a) Positive crossing



(a) Negative crossing

Figure 4.6: Octahedron at a crossing.

Applying Proposition 3.2.7, the cross-ratio at h^1 in Figure 4.6(a) is given by

$$\frac{c(h^2)c(e_m)}{\sigma(s^{42})c(h^4)c(e_j)} = \frac{c(e_m)}{\sigma(s^{42})c(e_j)} = \frac{c(e_m)}{\overline{\sigma}(\mu_\beta)c(e_j)}.$$

By the cross-ratio at h^1, we mean the cross-ratio at l_3 with respect to the tetrahedron chosen as in Figure 4.2. We use terms the cross-ratios at h^3, h^5, h_1, h_3, in a same manner. Similar computation gives us that the cross-ratios at h^1, h^3, h^5, h_1, h_3 for Figure 5.5(a) are respectively given by

$$\frac{c(e_m)}{\overline{\sigma}(\mu_\beta)c(e_j)}, \frac{\overline{\sigma}(\mu_\beta)c(e_k)}{c(e_l)}, \frac{c(e_j)c(e_l)}{c(e_m)c(e_k)}, \frac{c(e_k)}{\overline{\sigma}(\mu_\alpha)c(e_j)}, \frac{\overline{\sigma}(\mu_\alpha)c(e_m)}{c(e_l)}$$

and the cross-ratios at h^1, h^3, h^5, h_1, h_3 for Figure 4.6(b) are respectively given by

$$\frac{c(e_j)}{\overline{\sigma}(\mu_\alpha)c(e_k)}, \frac{\overline{\sigma}(\mu_\alpha)c(e_l)}{c(e_m)}, \frac{c(e_m)c(e_k)}{c(e_j)c(e_l)}, \frac{c(e_j)}{\overline{\sigma}(\mu_\beta)c(e_m)}, \frac{\overline{\sigma}(\mu_\beta)c(e_l)}{c(e_k)}.$$

The proposition directly follows from comparing the above cross-ratios with the cross-ratios given in Figure 4.2 and 4.3. We remark again that the above cross-ratios are non-degenerate and satisfy the gluing equations for $\mathcal{T}$. □

For a representation $\rho : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ we say that a cocycle $\sigma : \partial N^1 \to \mathbb{C}^\times$ is *associated to* ρ if

$$\rho|_\Sigma(\gamma) = \begin{pmatrix} \overline{\sigma}(\gamma) & * \\ 0 & \overline{\sigma}(\gamma)^{-1} \end{pmatrix}$$

up to conjugation for all $\gamma \in \pi_1(\Sigma)$ and for any component Σ of ∂N. Here $\rho|_\Sigma : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{C})$ means the restriction. Since every component Σ of ∂N is either a sphere or a torus, the restriction $\rho|_\Sigma$ is reducible. Therefore, for any representation ρ there exists a cocycle σ associated to ρ.

Theorem 4.2.1. Let $\rho : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ be a representation such that $\rho(\mu_i) \neq \pm I$ for all $1 \leqslant i \leqslant h$. Then for any cocycle $\sigma : \partial N^1 \to \mathbb{C}^\times$ associated to ρ, there exists a σ-deformed Ptolemy assignment c such that $\rho_c = \rho$ up to conjugation.

A proof of Theorem 4.2.1 is essentially also given in [CYZ18, §4] (see also [Cho16a]). The proof given in [CYZ18] assume that ρ is a (lifting of) $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation, but this is not actually required in the proof. For completeness of the paper, we present a detailed proof of Theorem 4.2.1 in Section 4.2.1.

Corollary 4.2.1 (Theorem 1.2.2)**.** Let $\rho : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation satisfying $\rho(\mu_i) \neq \pm I$ for all $1 \leqslant i \leqslant h$. If the representation ρ admits a $\mathrm{SL}(2,\mathbb{C})$-lifting, then there exists a non-degenerate solution $(\mathbf{w}, \mathbf{m})$ such that $\rho_{\mathbf{w},\mathbf{m}} = \rho$ up to conjugation.

Proof. For each sphere component Σ of ∂N, the restriction $\rho|_\Sigma : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{C})$ is clearly trivial. Thus one can choose an associated cocycle σ such that it is trivial on the sphere components. Then the proof directly follows from Proposition 4.2.1 and Theorem 4.2.1. $\square$

4.2.1 Proof of Theorem 4.2.1

For simplicity we may assume that a given cocycle $\sigma : \partial N^1 \to \mathbb{C}^\times$ is trivial on the sphere components. Let $\widetilde{N}$ be the universal cover of N. We lift σ to $\partial \widetilde{N}$, and denote the resulting cocycle also by $\sigma : \partial \widetilde{N}^1 \to \mathbb{C}^\times$.

Definition 4.2.2. A *decoration* $\mathcal{D} : \widetilde{N}^0 \to \mathbb{C}^2 \backslash \{(0,0)^t\}$ is an assignment satisfying

- (ρ-equivariance) $\mathcal{D}(\gamma \cdot v) = \rho(\gamma)\mathcal{D}(v)$ for all $\gamma \in \pi_1(N)$ and $v \in \widetilde{N}^0$;
- $\mathcal{D}(v_2) = \sigma(s)\mathcal{D}(v_1)$ for all $s \in \partial \widetilde{N}^1$ where v_1 and v_2 are the initial and terminal vertices of s, respectively.

Remark that a decoration exists, since a given cocycle σ is associated to ρ. For a decoration $\mathcal{D}$ we define $c : \mathcal{T}^1 \to \mathbb{C}$ by

$$c(e) = \det(\mathcal{D}(v_1), \mathcal{D}(v_2))$$

for $e \in \mathcal{T}^1$ where v_1 and v_2 are the initial and terminal vertices of any lifting of e, viewed as a long edge of N, respectively. Note that $c(e)$ does not depend on the choice of a lifting of e, since $\mathcal{D}$ is ρ-equivariant. Also, note that $c(-e) = -c(e)$ for all $e \in \mathcal{T}^1$.

Proposition 4.2.2. If $c(e) \neq 0$ for all $e \in \mathcal{T}^1$, then $c : \mathcal{T}^1 \to \mathbb{C}^\times$ is a σ-deforemd Ptolemy assignment.

Proof. Let us choose a lifting of an ideal triangulation Δ of $\mathcal{T}$. We denote the edges of its truncation as in Definition 4.2.1; l_i denotes a long-edge and s_{ij} denotes the short edge running from l_i to l_j. We also denote the initial and terminal vertices of l_i by v_i and v^i, respectively as in Figure 4.7.

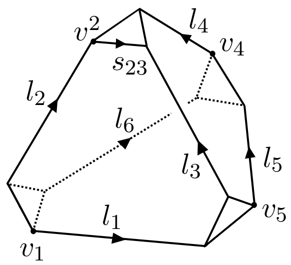


Figure 4.7: A truncated tetrahedron.

Applying the Plucker relation to $\mathcal{D}(v_1), \mathcal{D}(v_5), \mathcal{D}(v_4), \mathcal{D}(v^2)$, we obtain

$$\begin{aligned}&\det(\mathcal{D}(v_1), \mathcal{D}(v_4)) \det(\mathcal{D}(v_5), \mathcal{D}(v^2)) \\ &= \det(\mathcal{D}(v_1), \mathcal{D}(v_5)) \det(\mathcal{D}(v_4), \mathcal{D}(v^2)) + \det(\mathcal{D}(v_1), \mathcal{D}(v^2)) \det(\mathcal{D}(v_5), \mathcal{D}(v_4)).\end{aligned}$$

By construction of c, it is equivalent to

$$\begin{aligned}\sigma(s_{61})\sigma(s_{64})c(l_6)\,\sigma(s_{32})\sigma(s_{35})c(l_3) &= \sigma(s_{15})c(l_1)\sigma(s_{42})c(l_4) + \sigma(s_{21})c(l_2)\sigma(s_{54})c(l_5) \\ \Leftrightarrow\ c(l_3)c(l_6) &= \tfrac{\sigma(s_{23})}{\sigma(s_{35})}\tfrac{\sigma(s_{26})}{\sigma(s_{65})}c(l_2)c(l_5) + \tfrac{\sigma(s_{13})}{\sigma(s_{34})}\tfrac{\sigma(s_{16})}{\sigma(s_{64})}c(l_1)c(l_4).\end{aligned}$$

Therefore, $c : \mathcal{T}^1 \to \mathbb{C}^\times$ is a σ-deformed Ptolemy assignment. □

Therefore, it is enough to prove that there exists a decoration $\mathcal{D}$ such that the induced assignment $c : \mathcal{T}^1 \to \mathbb{C}$ satisfies $c(e) \neq 0$ for all $e \in \mathcal{T}^1$.

We first consider the regional edges $e_1, \cdots, e_n$ of $\mathcal{T}$. We choose a lifting, $\widetilde{e}_j$, of each e_j so that their terminal point agree as in Figure 4.8. Let v_k^0 and v_k^1 be the initial and terminal points of $\widetilde{e}_j$, viewed as an edge of $\widetilde{N}$, respectively. Since $\sigma : \partial N^1 \to \mathbb{C}^\times$ is trivial on the sphere components, we have $\mathcal{D}(v_j^1) = \mathcal{D}(v_k^1)$. Moreover, from ρ-equivariance of $\mathcal{D}$, we have

$$\mathcal{D}(v_j^0) = \rho(g)\mathcal{D}(v_k^0) \tag{4.2.4}$$

for some $g \in \pi_1(N)$. From elementary covering theory one can check that if

$e_j \cup e_k$ wraps an arc of K, then the loop g should be the Wirtinger generator corresponding to the arc; see Figure 4.8. For simplicity we let $W = \mathcal{D}(v_j^1)(= \mathcal{D}(v_k^1))$ and $V_j = \mathcal{D}(v_j^0)$ for $1 \leqslant j \leqslant m$. Note that $c(e_j) \neq 0$ if and only if $\det(W, V_j) \neq 0$.

We then consider the edges of $\mathcal{T}$ that intersect $\nu(L)$. Let us consider an ideal triangle (with edges denoted by x, y, e_k) in $S^3 \backslash (L \cup \{p,q\})$ together with its lifting (with edges denoted by $\widetilde{x}, \widetilde{y}, \widetilde{e}_k$) as in Figure 4.8. Let v_x and v_y be the initial vertices of $\widetilde{x}$ and $\widetilde{y}$, again viewed as edges of $\widetilde{N}$, respectively. Then for the Wirtinger generator g, we have

$$\rho(g)\mathcal{D}(v_x) = \mathcal{D}(g \cdot v_x) = \overline{\sigma}(g)^{\pm 1}\mathcal{D}(v_x).$$

Therefore, $\mathcal{D}(v_x)$ is an eigenvector of $\rho(g)$. It follows that $c(x) = \det(W, \mathcal{D}(v_x)) \neq 0$ if and only if W is not an eigenvector of $\rho(g)$. Similarly, $c(y) \neq 0$ if and only if V_k is not an eigenvector of $\rho(g)$.

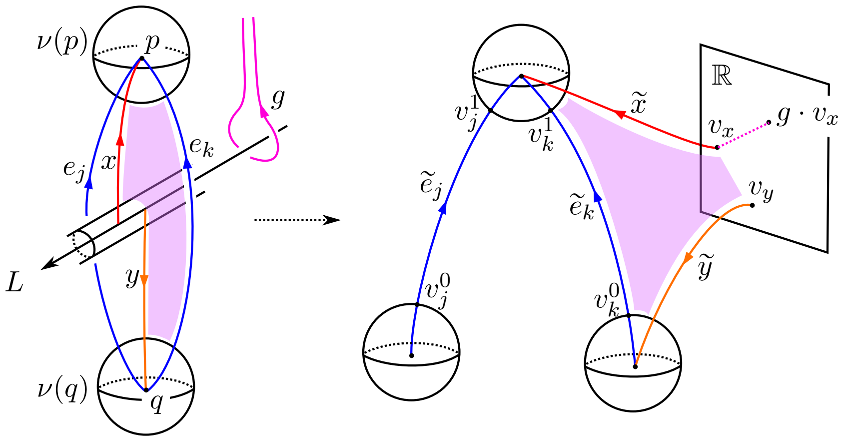


Figure 4.8: Local configuration of a lifting.

We finally consider an edge of $\mathcal{T}$ that joins q to itself. Let us consider an ideal triangle (with edges denoted by e_j, e_k, z) in $S^3 \backslash (L \cup \{p,q\})$ together with its lifting (with edges denoted by $\widetilde{e}_j, \widetilde{e}_k, \widetilde{z}$) as in Figure 4.9. It follows that $c(z) \neq 0$

if and only if $\det(V_j, V_k) = \det(\rho(g)V_k, V_k) \neq 0$ (recall the equation (4.2.4)). It is equivalent to the condition that V_k is not an eigenvector of $\rho(g)$. Similarly, for an edge z of $\mathscr{T}$ that joins p to itself, we conclude that $c(z) \neq 0$ if and only if W is not an eigenvector of $\rho(g)$.

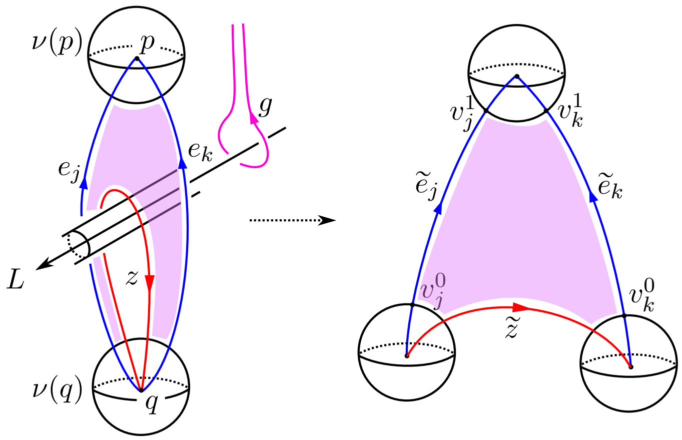


Figure 4.9: Local configuration of a lifting.

Let us sum up the required conditions. To be precise, we enumerate the Wirtinger generators by $g_1, \cdots, g_l$. A desired decoration should satisfy (i) $\det(W, V_j) \neq 0$; (ii) W is not an eigenvector of $\rho(g_i)$; (iii) V_j is not an eigenvector of $\rho(g_i)$ for all $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant l$. Since we can choose W and one of V_j's freely, such a decoration exists. See, for instance, Lemma 2.1 in [Cho16a]. See also Examples 4.3.1 and 4.3.2.

4.3 Complex volume formula

We devote this section to prove Theorem 1.2.3. For convenience of the reader, let us recall the theorem.

We fix a meridian μ_i and let λ_i be the canonical longitude of each component of a link L. For $\kappa = (r_1, s_1, \cdots, r_h, s_h)$ we denote by M_κ the manifold obtained from M by Dehn-filling that kills the curve $r_j\mu_j + s_j\lambda_j$ on each boundary torus

Σ_j, where (r_j, s_j) is either a pair of coprime integers or the symbol ∞ meaning that we do not fill the corresponding boundary torus.

Let $\rho : \pi_1(M_\kappa) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation. If M_κ has non-empty boundary, we assume that ρ is a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation so that the volume and Chern-Simons invariant of ρ are well-defined. Regarding ρ as a representation from $\pi_1(M)$ by compositing the inclusion $\pi_1(M) \to \pi_1(M_\kappa)$, we have

$$\begin{cases} \mathrm{tr}(\rho(\mu_i)) = \pm 2,\ \mathrm{tr}(\rho(\lambda_i)) = \pm 2 & \text{for } (r_i, s_i) = \infty \\ \rho(\mu_i^{r_i}\lambda_i^{s_i}) = \pm I & \text{for } (r_i, s_i) \neq \infty \end{cases} \tag{4.3.5}$$

where r_i and s_i are coprime integers. If we assume that $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ admits a $\mathrm{SL}(2,\mathbb{C})$-lifting and $\rho(\mu_i) \neq \pm I$ for all $1 \leqslant i \leqslant h$, then there exists a point $(\mathbf{w}, \mathbf{m})$ such that $\rho_{\mathbf{w},\mathbf{m}} = \rho$ up to conjugation where m_i is an eigenvalue of $\rho(\mu_i)$. Recall Corollary 4.2.1 and Theorem 3.3.1. It follows from the equation (4.3.5) that for $\kappa_i \neq \infty$ we have $m_i^{r_i} l_i^{s_i} = \pm 1$ and thus $r_i \log m_i + s_i \log l_i \equiv 0$ in modulo πi where l_i is an eigenvalue $\rho(\lambda_i)$. From coprimeness of (r_i, s_i), there exists integers u_i and v_i satisfying

$$r_i \log m_i + s_i \log l_i + \pi i (r_i u_i + s_i v_i) = 0. \tag{4.3.6}$$

Theorem 4.3.1 (Theorem 1.2.3)**.** The complex volume of $\rho : \pi_1(M_\kappa) \to \mathrm{PSL}(2,\mathbb{C})$ is given by

$$i\mathrm{Vol}_{\mathbb{C}}(\rho) \equiv \mathbb{W}_0(\mathbf{w}, \mathbf{m}) \mod \pi^2\mathbb{Z}$$

where the function $\mathbb{W}_0(w_1, \cdots, w_n, m_1, \cdots, m_h)$ is defined by

$$\begin{aligned} \mathbb{W}_0 := \mathbb{W}(w_1, \cdots, w_n, m_1, \cdots, m_h) - \sum_{j=1}^{n} \left(w_j \frac{\partial \mathbb{W}}{\partial w_j} \right) \log w_j \\ - \sum_{(r_i, s_i) \neq \infty} \left[\left(m_i \frac{\partial \mathbb{W}}{\partial m_i} \right) (\log m_i + u_i \pi i) - \frac{r_i}{s_i} (\log m_i + u_i \pi i)^2 \right]. \end{aligned}$$

4.3.1 Proof of Theorem 4.3.1

We assign a vertex-ordering of each tetrahedron Δ of $\mathcal{T}$ as in Figure 5.5. Note that these orderings agree on the common faces, so we may orient every edge of $\mathcal{T}$ with respect to this vertex-ordering. We say that Δ is *positively oriented* if the orientation of Δ induced from the vertex-ordering agrees with the orientation of N, and Δ is *negatively oriented*, otherwise. We let $\epsilon_\Delta = \pm 1$ according to this orientation of Δ.

Let $\widetilde{\rho} : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ be a lifting of ρ and $\sigma : \partial N^1 \to \mathbb{C}^\times$ be a cocycle assoicated to $\widetilde{\rho}$ which is trivial on the sphere components. From the equation (4.3.5) we have

$$\begin{cases} \overline{\sigma}(\mu_i) = \pm 1,\ \overline{\sigma}(\lambda_i) = \pm 1 & \text{for } (r_i, s_i) = \infty \\ \overline{\sigma}(\mu_i^{r_i} \lambda_i^{s_i}) = \pm 1 & \text{for } (r_i, s_i) \neq \infty \end{cases}.$$

Recall Proposition 3.3.1 that there exists a cocycle $a : \partial N^1 \to \mathbb{C}$ such that (i) $a(e) \equiv \log \sigma(e)$ in modulo $\pi i \mathbb{Z}$ for all $e \in \partial N^1$; (ii) a is trivial on the sphere components; (iii) the induced homomorphism $\overline{a}$ satisfies

$$\begin{cases} \overline{a}(\mu_i) = \overline{a}(\lambda_i) = 0 & \text{for } (r_i, s_i) = \infty \\ \overline{a}(\mu_i) = \log \overline{\sigma}(\mu_i) + u_i \pi i \text{ and } \overline{a}(\lambda_i) = \log \overline{\sigma}(\lambda_i) + v_i \pi i & \text{for } (r_i, s_i) \neq \infty \end{cases}.$$

The equation (4.3.6) tells us that $r_i \overline{a}(\mu_i) + s_i \overline{a}(\lambda_i) = 0$ for all $\kappa_i \neq \infty$. On the other hand, by Theorem 4.2.1 there exists a σ-deformed Ptolemy assignment $c : \mathcal{T}^1 \to \mathbb{C}^\times$ such that $\rho_c = \widetilde{\rho}$ up to conjugation. We let

$$(\mathbf{w}, \mathbf{m}) = (c(e_1), \cdots, c(e_n), \overline{\sigma}(\mu_1), \cdots, \overline{\sigma}(\mu_h))$$

as in Proposition 4.2.1.

For each ideal tetrahedron Δ (with edges denoted as in Figure 3.3) of $\mathcal{T}$, we let

$$\begin{aligned}
z &= \frac{\sigma(s_{12})\sigma(s_{45})}{\sigma(s_{24})\sigma(s_{51})}\frac{c(l_1)c(l_4)}{c(l_2)c(l_5)} \\
p\pi i &= (a(s_{12}) + a(s_{45}) - a(s_{24}) - a(s_{51}) \\
&\quad + \log c(l_1) + \log c(l_4) - \log c(l_2) - \log c(l_5) - \log z \\
q\pi i &= a(s_{53}) + a(s_{26}) - a(s_{32}) - a(s_{65}) \\
&\quad + \log c(l_2) + \log c(l_5) - \log c(l_3) - \log c(l_6) + \log(1-z)
\end{aligned}$$

if $\epsilon_\Delta = 1$ and

$$\begin{aligned}
z &= \frac{\sigma(s_{24})\sigma(s_{51})}{\sigma(s_{12})\sigma(s_{45})}\frac{c(l_2)c(l_5)}{c(l_1)c(l_4)} \\
p\pi i &= a(s_{24}) + a(s_{51}) - a(s_{12}) - a(s_{45}) \\
&\quad + \log c(l_2) + \log c(l_5) - \log c(l_1) - \log c(l_4) - \log z \\
q\pi i &= a(s_{43}) + a(s_{16}) - a(s_{64}) - a(s_{31}) \\
&\quad + \log c(l_1) + \log c(l_4) - \log c(l_3) - \log c(l_6) + \log(1-z)
\end{aligned}$$

if $\epsilon_\Delta = -1$. We let $R(\Delta) := R(z; p, q)$ where R is the extended Rogers dilogarithm given by

$$R(z; p, q) = \mathrm{Li}_2(z) + \frac{\pi i}{2}(p\log(1-z) + q\log z) + \frac{1}{2}\log(1-z)\log z - \frac{\pi^2}{2}.$$

Theorem 3.3.1 gives that

$$i\mathrm{Vol}_{\mathbb{C}}(\rho) \equiv \sum_{\Delta} \epsilon_\Delta R(\Delta) \mod \pi^2\mathbb{Z} \tag{4.3.7}$$

where the sum is over all tetrahedra Δ of $\mathcal{T}$. We refer to Chapter 3 for details. Therefore, it is enough to show that the right-hand side of the equation (4.3.7) is equal to $\mathbb{W}_0(\mathbf{w}, \mathbf{m})$ in modulo $\pi^2\mathbb{Z}$.

Let us first consdier a crossing of L as in Figure 4.1(a). At this crossing, we denote edges of $\mathscr{T}$ as in Figure 5.5(a). We also denote by Δ^1 the tetrahedron corresponding to the edge h^1 as in Figure 4.2, and denote similarly for h^3, h^5, h_1, and h_3. It is not hard to check that $\epsilon_{\Delta^1} = \epsilon_{\Delta^5} = \epsilon_{\Delta_1} = 1$ and $\epsilon_{\Delta^3} = \epsilon_{\Delta_3} = -1$. A straightforward computation gives

$$\begin{aligned}R(\Delta^1) &= \mathrm{Li}_2(\tfrac{w_m}{m_\beta w_j}) - \tfrac{\pi^2}{6} + \tfrac{1}{2}(\log w_m - \log w_j - \overline{a}(\mu_\beta))\log(1 - \tfrac{w_m}{m_\beta w_j}) \\ &+ \tfrac{1}{2}(\log w_j - \log c(h^5) + \log c(h^2) - \log c(h^1) + a(s^{41}) + \log(1 - \tfrac{w_m}{m_\beta w_j}))\log \tfrac{w_m}{m_\beta w_j}.\end{aligned}$$

Since $\log \frac{w_m}{m_\beta w_j} \equiv \log w_m - \log w_j - \overline{a}(\mu_\beta)$ in modulo $2\pi i$,

$$\begin{aligned}R(\Delta^1) &\equiv \mathrm{Li}_2(\tfrac{w_m}{m_\beta w_j}) - \tfrac{\pi^2}{6} + (\log w_m - \log w_j - \overline{a}(\mu_\beta))\log(1 - \tfrac{w_m}{m_\beta w_j}) \\ &+ \tfrac{1}{2}(\log w_j - \log c(h^5) + \log c(h^2) - \log c(h^1) + a(s^{41}))(\log w_m - \log w_j - \overline{a}(\mu_\beta))\end{aligned}$$

in modulo $\pi^2\mathbb{Z}$. We similarly compute the Rogers dilogarithm terms for other tetrahedra and obtain :

$$\begin{aligned}&R(\Delta^1) - R(\Delta^3) + R(\Delta_1) - R(\Delta_3) + R(\Delta^5) \\ &= \mathrm{Li}_2(\tfrac{w_m}{m_\beta w_j}) - \mathrm{Li}_2(\tfrac{w_l}{m_\beta w_k}) + \mathrm{Li}_2(\tfrac{w_k}{m_\alpha w_j}) - \mathrm{Li}_2(\tfrac{w_l}{m_\alpha w_m}) + \mathrm{Li}_2(\tfrac{w_j w_l}{w_k w_m}) - \tfrac{\pi^2}{6} \\ &+ (\log w_m - \log w_j - \overline{a}(\mu_\beta))\log(1 - \tfrac{w_m}{m_\beta w_j}) \\ &+ (\log w_k - \log w_l + \overline{a}(\mu_\beta))\log(1 - \tfrac{w_l}{m_\beta w_k}) \\ &+ (\log w_k - \log w_j - \overline{a}(\mu_\alpha))\log(1 - \tfrac{w_k}{m_\alpha w_j}) \\ &+ (\log w_m - \log w_l + \overline{a}(\mu_\alpha))\log(1 - \tfrac{w_l}{m_\alpha w_m}) \\ &+ (\log w_l + \log w_j - \log w_k - \log w_m)\log(1 - \tfrac{w_j w_l}{w_k w_m}) \\ &+ \tfrac{1}{2}(\log w_j - \log c(h^5) + \log c(h^2) - \log c(h^1) + a(s^{41}))(\log w_m - \log w_j - \overline{a}(\mu_\beta)) \\ &+ \tfrac{1}{2}(\log w_k - \log c(h^5) + \log c(h^2) - \log c(h^3) + a(s^{43}))(\log w_k - \log w_l + \overline{a}(\mu_\beta)) \\ &+ \tfrac{1}{2}(\log w_j - \log c(h_5) + \log c(h_2) - \log c(h_1) + a(s_{41}))(\log w_k - \log w_j - \overline{a}(\mu_\alpha))\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}(\log w_m-\log c(h_5)+\log c(h_2)-\log c(h_3)+a(s_{43}))(\log w_m-\log w_l+\overline{a}(\mu_\alpha))\\
&+\tfrac{1}{2}(\log w_k+\log w_m-\log c(h^5)-\log c(h_5))(\log w_l+\log w_j-\log w_k-\log w_m)
\end{aligned}
$$

Rearranging the last five lines appropriately, we obtain

$$
\begin{aligned}
&R(\Delta^1)-R(\Delta^3)+R(\Delta_1)-R(\Delta_3)+R(\Delta_5)\\
&\left.\begin{aligned}
&=\mathrm{Li}_2(\tfrac{w_m}{m_\beta w_j})-\mathrm{Li}_2(\tfrac{w_l}{m_\beta w_k})+\mathrm{Li}_2(\tfrac{w_k}{m_\alpha w_j})-\mathrm{Li}_2(\tfrac{w_l}{m_\alpha w_m})+\mathrm{Li}_2(\tfrac{w_j w_l}{w_k w_m})-\tfrac{\pi^2}{6}\\
&+(\log w_m-\log w_j-\overline{a}(\mu_\beta))\log(1-\tfrac{w_m}{m_\beta w_j})\\
&+(\log w_k-\log w_l+\overline{a}(\mu_\beta))\log(1-\tfrac{w_l}{m_\beta w_k})\\
&+(\log w_k-\log w_j-\overline{a}(\mu_\alpha))\log(1-\tfrac{w_k}{m_\alpha w_j})\\
&+(\log w_m-\log w_l+\overline{a}(\mu_\alpha))\log(1-\tfrac{w_l}{m_\alpha w_m})\\
&+(\log w_l+\log w_j-\log w_k-\log w_m)\log(1-\tfrac{w_j w_l}{w_k w_m})\\
&-(\log w_m-\log w_j-\overline{a}(\mu_\beta))(\log w_k-\log w_j-\overline{a}(\mu_\alpha))
\end{aligned}\right\}\text{A-part}\\
&\left.\begin{aligned}
&+\tfrac{1}{2}\log c(h^2)\,(\log w_m+\log w_k-\log w_l-\log w_j)\\
&+\tfrac{1}{2}\log c(h_2)\,(\log w_m+\log w_k-\log w_l-\log w_j)\\
&-\tfrac{1}{2}\log c(h^1)\,(\log w_m-\log w_j-\overline{a}(\mu_\beta))\\
&-\tfrac{1}{2}\log c(h^3)\,(\log w_k-\log w_l+\overline{a}(\mu_\beta))\\
&-\tfrac{1}{2}\log c(h_1)\,(\log w_k-\log w_j-\overline{a}(\mu_\alpha))\\
&-\tfrac{1}{2}\log c(h_3)\,(\log w_m-\log w_l+\overline{a}(\mu_\alpha))
\end{aligned}\right\}\text{B-part}\\
&\left.\begin{aligned}
&+\tfrac{1}{2}a(s^{41})(\log w_m-\log w_j)+\tfrac{1}{2}a(s^{43})(\log w_k-\log w_l)\\
&+\tfrac{1}{2}a(s_{21})(\log w_k-\log w_j)+\tfrac{1}{2}a(s_{23})(\log w_m-\log w_l)
\end{aligned}\right\}\text{C-part}\\
&\left.+\overline{a}(\mu_\alpha)\overline{a}(\mu_\beta)-\tfrac{1}{2}a(s^{31})\overline{a}(\mu_\beta)-\tfrac{1}{2}a(s_{31})\overline{a}(\mu_\alpha)\right\}\text{D-part}\\
&\left.+\tfrac{1}{2}\overline{a}(\mu_\alpha)(\log w_k-\log w_l)+\tfrac{1}{2}\overline{a}(\mu_\beta)(\log w_j-\log w_k)\right\}\text{E-part}
\end{aligned}
$$

Note that in the above computation, $\log c(h^5)$- and $\log c(h_5)$-terms vanish, and we replace $a(s_{41})$ and $a(s_{43})$ by $\overline{a}(\mu_\alpha)+a(s_{21})$ and $\overline{a}(\mu_\alpha)+a(s_{23})$, respectively. We compute similarly for a crossing as in Figure 4.1(b) and obtain:

$$
\begin{aligned}
&-R(\Delta^1)+R(\Delta^3)-R(\Delta_1)+R(\Delta_3)-R(\Delta^5)\\
&\left.\begin{aligned}
&=-\mathrm{Li}_2(\tfrac{m_\beta w_m}{w_j})+\mathrm{Li}_2(\tfrac{m_\beta w_l}{w_k})-\mathrm{Li}_2(\tfrac{m_\alpha w_k}{w_j})+\mathrm{Li}_2(\tfrac{m_\alpha w_l}{w_m})-\mathrm{Li}_2(\tfrac{w_j w_l}{w_k w_m})+\tfrac{\pi^2}{6}\\
&-(\log w_m-\log w_j+\overline{a}(\mu_\beta))\log(1-\tfrac{m_\beta w_m}{w_j})\\
&-(\log w_k-\log w_l-\overline{a}(\mu_\beta))\log(1-\tfrac{m_\beta w_l}{w_k})\\
&-(\log w_k-\log w_j+\overline{a}(\mu_\alpha))\log(1-\tfrac{m_\alpha w_k}{w_j})\\
&-(\log w_m-\log w_l-\overline{a}(\mu_\alpha))\log(1-\tfrac{m_\alpha w_l}{w_m})\\
&-(\log w_l+\log w_j-\log w_k-\log w_m)\log(1-\tfrac{w_j w_l}{w_k w_m})\\
&+(\log w_m-\log w_j+\overline{a}(\mu_\beta))(\log w_k-\log w_j+\overline{a}(\mu_\alpha))
\end{aligned}\right\}\text{A-part}\\
&\left.\begin{aligned}
&-\tfrac{1}{2}\log c(h^2)\,(\log w_m+\log w_k-\log w_l-\log w_j)\\
&-\tfrac{1}{2}\log c(h_2)\,(\log w_m+\log w_k-\log w_l-\log w_j)\\
&+\tfrac{1}{2}\log c(h_1)\,(\log w_m-\log w_j+\overline{a}(\mu_\beta))\\
&+\tfrac{1}{2}\log c(h_3)\,(\log w_k-\log w_l-\overline{a}(\mu_\beta))\\
&+\tfrac{1}{2}\log c(h^1)\,(\log w_k-\log w_j+\overline{a}(\mu_\alpha))\\
&+\tfrac{1}{2}\log c(h^3)\,(\log w_m-\log w_l-\overline{a}(\mu_\alpha))
\end{aligned}\right\}\text{B-part}\\
&\left.\begin{aligned}
&-\tfrac{1}{2}a(s^{41})(\log w_k-\log w_j)-\tfrac{1}{2}a(s^{43})(\log w_m-\log w_l)\\
&-\tfrac{1}{2}a(s_{21})(\log w_m-\log w_j)-\tfrac{1}{2}a(s_{23})(\log w_l-\log w_k)
\end{aligned}\right\}\text{C-part}\\
&-\overline{a}(\mu_\alpha)\overline{a}(\mu_\beta)-\tfrac{1}{2}a(s^{31})\overline{a}(\mu_\alpha)-\tfrac{1}{2}a(s_{31})\overline{a}(\mu_\beta)\ \Big\}\text{D-part}\\
&+\tfrac{1}{2}\overline{a}(\mu_\beta)(\log w_j-\log w_k)+\tfrac{1}{2}\overline{a}(\mu_\alpha)(\log w_k-\log w_l)\ \Big\}\text{E-part}
\end{aligned}
$$

As one can see, we divide the Rogers dilogarithm terms coming from a crossing into 5 parts: A, B, C, D, and E-parts.

Let us first consider A-parts. If we use the equality

$$\begin{aligned}
&-(\log w_k - \log w_j - \overline{a}(\mu_\alpha))(\log w_m - \log w_j - \overline{a}(\mu_\beta)) \\
&= - \left(\log w_k - \log w_j - \overline{a}(\mu_\alpha) - \log \tfrac{w_k}{m_\alpha w_j}\right)(\log w_m - \log w_j - \overline{a}(\mu_\beta)) \\
&\qquad - \log \tfrac{w_k}{m_\alpha w_j}(\log w_m - \log w_j - \overline{a}(\mu_\beta)) \\
&\equiv - \left(\log w_k - \log w_j - \overline{a}(\mu_\alpha) - \log \tfrac{w_k}{m_\alpha w_j}\right)\log \tfrac{w_m}{m_\beta w_j} \\
&\qquad - \log \tfrac{w_k}{m_\alpha w_j}(\log w_m - \log w_j - \overline{a}(\mu_\beta)) \qquad\qquad (\mathrm{mod}\ \pi^2\mathbb{Z}),
\end{aligned}$$

then one can directly check that the sum of A-parts over all crossings is equal to

$$\mathbb{W}(\mathbf{w}, \mathbf{m}) - \sum_{j=1}^{n}\left(w_j \frac{\partial \mathbb{W}}{\partial w_j}\right)\log w_j - \sum_{i=1}^{h}\left(m_i \frac{\partial \mathbb{W}}{\partial m_i}\right)\overline{a}(\mu_i).$$

For D-parts, the sum of $-\frac{1}{2}a(s^{31})\overline{a}(\mu_i)$-terms along the i-th component of L results in $-\frac{1}{2}\overline{a}(\lambda_{i;bf})\overline{a}(\mu_i)$, where $\lambda_{i;bf}$ is the blackboard framed longitude of the i-th component. Similarly, the sum of $-\frac{1}{2}a(s_{31})b_i(\mu_i)$-terms also results in $-\frac{1}{2}\overline{a}(\lambda_{i;bf})\overline{a}(\mu_i)$. The remaining terms $\pm\overline{a}(\mu_i)\overline{a}(\mu_j)$ revise the framing appropriately and so the sum of D-parts over all crossings is equal to

$$-\sum_{i=1}^{h}\overline{a}(\mu_i)\overline{a}(\lambda_i).$$

Lemma 4.3.1. The sum of B-parts over all crossings vanishes.

Proof. Let e be an over edge of $\mathcal{T}$ with the corresponding over-arc of L as in Figure 4.4(a). Note that the edge e appears as h_1 at the initial crossing, as h_3 at the terminal crossing, and as $h^2 = h^4$ at the intermediate crossings. Then, in the sum of B-parts, $\log c(e)$-terms appear exactly at these crossings and their sum is given by

$$\tfrac{1}{2}\log c(e)\Big[(-\log w_{j_1} + \log w_{j_2} - \overline{a}(\mu_i))$$

$$
\begin{aligned}
&+ (\log w_{j_1} - \log w_{j_2} - \log w_{j_3} + \log w_{j_4}) + \cdots \\
&+ (\log w_{j_{2m-1}} - \log w_{j_{2m}} - \log w_{j_{2m+1}} + \log w_{j_{2m+2}}) \\
&+ (\log w_{j_{2m+1}} - \log w_{j_{2m+2}} + \overline{a}(\mu_i))\Big] = 0.
\end{aligned}
$$

Note that changing orientations that are not specified in the local diagram dose not change the computation. We compute similarly for an under edge of $\mathscr{T}$, and complete the proof. □

We omit a proof the fact that the sum of D-parts and E-parts are respectively zero, since it can be checked combinatorially as in Lemma 4.3.1.

Recall that we have $\overline{a}(\mu_i) = 0$ for $\kappa_i = \infty$ and $r_i\overline{a}(\mu_i) + s_i\overline{a}(\lambda_i) = 0$ for $\kappa_i \neq \infty$. It thus follows that the sum of A- and D-parts over all crossings is equal to $\mathbb{W}_0(\mathbf{w}, \mathbf{m})$. This completes the proof, since the sums of B-, C-, and E-parts are all zero.

Example 4.3.1. We consider a diagram of the figure-eight knot and denote the Wirtinger generators by $g_1, \cdots, g_4$ as in Figure 4.10. It is known that

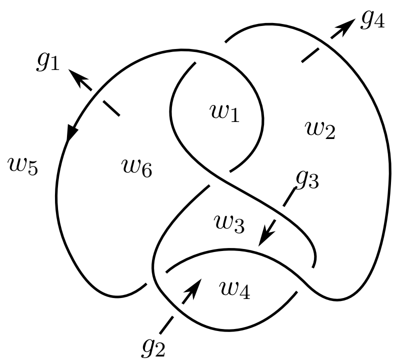


Figure 4.10: The figure eight knot diagram.

$$
\rho(g_1) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \text{ and } \rho(g_4) = \begin{pmatrix} m & 0 \\ y & m^{-1} \end{pmatrix}
$$

determine a -representation ρ of the knot group if

$$y = \frac{-m^4 + 3m^2 - 1 + \sqrt{m^8 - 2m^6 - m^4 - 2m^2 + 1}}{2m^2}.$$

The canonical longitude λ of the knot is given by $g_2\, g_4^{-1}\, g_2^{-1}\, g_1$, so an eignvalue l of $\rho(\lambda)$ is given by

$$l = \frac{m^8 - m^6 + 2m^4 - m^2 + 1 + (m^4 - 1)\sqrt{m^8 - 2m^6 - m^4 - 2m^2 + 1}}{2m^4}.$$

If we consider the $\frac{2}{3}$-Dehn filling, then we require $m \in \mathbb{C}^\times$ satisfying $m^2 l^3 = 1$; using the Mathematica, we have

$$(m, l) = (-1.30664 + 0.0498758i,\ -0.436423 + 0.713371i).$$

We remark that the representation ρ is in fact (a lifting of) the geometric representation for the $\frac{2}{3}$-filled manifold $M_{\frac{2}{3}}$ obtained from the figure-eight knot exterior. We let $(u, v) = (-2, 0)$ so that

$$2 \log m + 3 \log l + \pi i (2u + 3v) = 0.$$

We now consider the vectors V_j's, each of which corresponds to a region, as in Section 4.2.1. Recall that these vectors satisfy the condition

$$V_j = \rho(g_k)^{-1} V_i$$

at each arc as in Figure 4.11. (cf. region coloring in [CKS01, Cho16a].) Note that they are well-determined whenever an initial vector is chosen arbitrarily.

For instance, if we choose $V_6 = \binom{1}{i}$, then we have

$$V_1 = \begin{pmatrix} -0.847954 - 1.60327i \\ -0.448632 - 0.0566798i \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1.04988 + 1.30664i \\ 0.589028 + 0.0516843i \end{pmatrix},$$
$$V_3 = \begin{pmatrix} -0.784704 + 0.372425i \\ -0.392082 - 1.12719i \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0.61054 - 0.261719i \\ 1.12129 + 1.96967i \end{pmatrix},$$
$$V_5 = \begin{pmatrix} -0.764207 - 1.02917i \\ -0.0498758 - 1.30664i \end{pmatrix}, \quad V_6 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

We also choose another vector W almost arbitrarily; for instance, we let $W = \binom{2}{1}$. Then we have $\mathbf{w} = (w_1, \cdots, w_6)$ by $w_j = \det(W, V_j)$:

$$\begin{aligned} w_1 &= -0.0493091 + 1.48991i, & w_2 &= 0.12818 - 1.20327i, \\ w_3 &= 0.000538775 - 2.62681i, & w_4 &= 1.63204 + 4.20107i, \\ w_5 &= 0.664455 - 1.58411i, & w_6 &= -1 + 2i. \end{aligned}$$

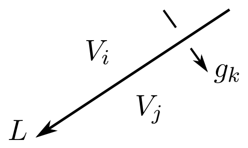


Figure 4.11: Rule for a region coloring.

Plugging the above non-degenerate solution $(\mathbf{w}, \mathbf{m}) = (w_1, \cdots, w_6, m)$ to Theorem 4.2.1, we obtain

$$i\mathrm{Vol}_{\mathbb{C}}(M_{\frac{2}{3}}) = -3.33835687 + 1.73712388i.$$

Note that changing choices for V_6 and V_0 may give a different non-degenerate solution but it results in the same volume and Chern-Simons invariant.

Example 4.3.2. Let us consider a diagram of the Whitehead link as in Figure

4.12. One can check that

$$\rho(g_1) = \begin{pmatrix} m_1 & 1 \\ 0 & m_1^{-1} \end{pmatrix} \text{ and } \rho(g_2) = \begin{pmatrix} m_2 & 0 \\ y & m_2^{-1} \end{pmatrix}$$

determine a -representation of the link group if

$$\begin{aligned} &m_1m_2(m_1^2-1)(m_2^2-1) + ((m_1^2m_2^2+1)(m_1^2-1)(m_2^2-1) + 2m_1^2m_2^2)y \\ &+ (2 - m_1^2 - m_2^2 + 2m_1^2m_2^2)m_1m_2y^2 + m_1^2m_2^2y^3 = 0. \end{aligned}$$

The longitude of the circular component is given by $g_5g_2^{-1}$ and that of the other

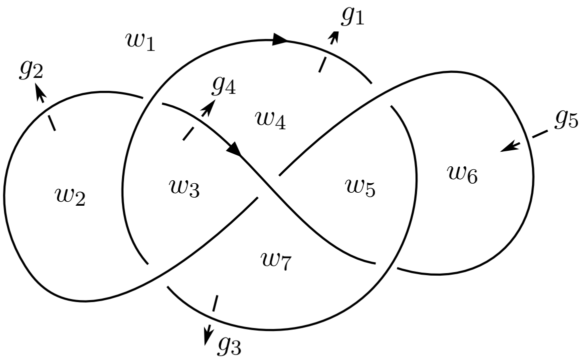


Figure 4.12: The Whitehead link.

component is given by $g_2g_1^{-1}g_3g_4^{-1}$. We obtain

$$\begin{aligned} l_1 = \tfrac{1}{m_1^2m_2^3}\Big[& m_1^4m_2(m_2^2-1)^2y^2 + m_1^3(m_2^2-1)(2m_2^2(y^2+1)-1)y \\ & + m_1^2(-m_2^5y^2 + m_2^3(y^4+5y^2+1) - 3m_2y^2) \\ & - m_1y(m_2^4(m_2^2+1) - 2m_2^2(y^2+1)+1) - m_2(m_2^2-1)y^2\Big], \\ l_2 = \tfrac{1}{m_1^3m_2}\Big[& m_1^4m_2^2y + m_1^3\left(-m_2^3y^2 + m_2y^2 + m_2\right) \\ & + m_1^2\left(-m_2^2y^3 - 2m_2^2y + y\right) + m_1m_2\left(m_2^2-2\right)y^2 + \left(m_2^2-1\right)y\Big]. \end{aligned}$$

Let us consider $\kappa = (-5, -\frac{5}{2})$ filling; using Mathematica, one can check that

$$(m_1, l_1) = (0.6043082 + 1.35916778i,\ 6.31524591 - 3.62462234i)$$
$$(m_2, l_2) = (1.4324890 + 1.08046977i,\ -4.30814400 - 0.19295781i)$$

satisfies (numerically) $m_i^{r_i} l_i^{s_i} = 1$ for $i = 1, 2$. We let $(u_1, v_1) = (0, 2)$ and $(u_2, v_2) = (-1, -1)$ so that the equation (4.3.6) holds for $i = 1, 2$.

Choosing an initial vector $V_1 = \binom{1}{i}$ and $W = \binom{2}{1}$, we obtain :

$$\begin{aligned}
w_1 &= -1 + 2i, & w_2 &= 1.93846759 - 5.78498860i,\\
w_3 &= -3.05190667 - 3.60341709i, & w_4 &= 0.62430373 - 1.81290671i,\\
w_5 &= -0.59085068 - 0.74757228i, & w_6 &= -1.23298500 + 2.38516959i,\\
w_7 &= -4.06836742 - 1.29382141i
\end{aligned}$$

Plugging the above non-degenerate solution $(\mathbf{w}, \mathbf{m}) = (w_1, \cdots, w_7, m_1, m_2)$ to Theorem 4.2.1, we obtain

$$i\mathrm{Vol}_{\mathbb{C}}(M_\kappa) = 1.185202630500 + 0.942707362517i.$$

Chapter 5

Cluster variables

Given a braid presentation D of a hyperbolic knot, Hikami and Inoue consider a system of polynomial equations arising from a sequence of cluster mutations determined by D. They show that any solution gives rise to shape parameters and thus determines a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$-representation of the knot group. They conjecture the existence of a solution corresponding to the geometric representation. In this paper, we show that a boundary-parabolic representation ρ arises from a solution if and only if the length of D modulo 2 equals the obstruction to lifting ρ to a boundary-parabolic -representation (as an element in $\mathbb{Z}_2$). In particular, the Hikami-Inoue conjecture holds if and only if the length of D is odd.

5.1 The Hikami-Inoue cluster variables

5.1.1 The octahedral decomposition

Let $K \subset S^3$ be a knot and let $\nu(K \cup \{p,q\})$ denote a tubular neigborhood of the union of K with two points $p \neq q \in S^3$ not in K. Whenever we choose a knot diagram representing K, we have a decomposition of the space $M = S^3 \backslash \nu(K \cup \{p,q\})$ into blocks each of which is a cube with two cylinders (whose core is the knot) removed. See Figure 5.1. Note that M is a 3-manifold with 3 boundary components (two spheres and a torus) whose interior is homeomorphic to $S^3 \backslash (K \cup \{p,q\})$. Now consider two quadrilaterals Q_1 and Q_2 in each block as in Figure 5.1 and collapse them horizontally so that their vertical edges are respectively identified. We call the resulting object a *pinched block*.

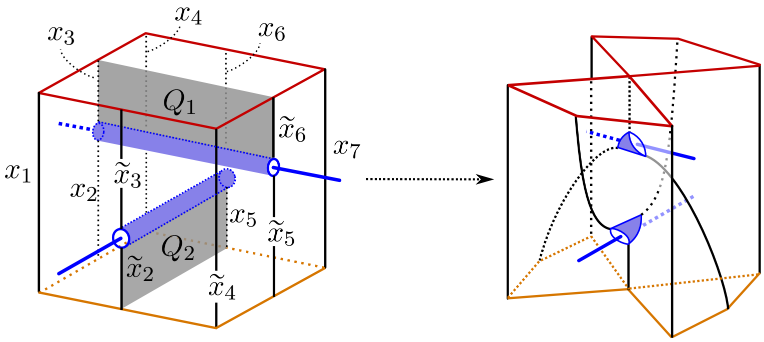


Figure 5.1: A pinched block

On the other hand, a pinched block can also be obtained from a truncated octahedron by identifying two pairs of edges as in Figure 5.2 (right). Therefore, one can obtain M by gluing truncated octahedra, and it thus follows that the interior of M can be decomposed into ideal octahedra (one per crossing). We denote this octahedral decomposition of $S^3 \backslash (K \cup \{p,q\})$ by $\mathcal{O}$. It is due to Dylan Thuston [Thu99] (see also [Wee05]).

5.1.2 The Hikami-Inoue cluster variables

An ideal octahedron as in Figure 5.2 has 12 edges each of which corresponds to a vertical edge of a cube in Figure 5.1. We may label those edges by $x_1, \cdots, x_7, \widetilde{x}_1, \cdots, \widetilde{x}_7$ as in Figure 5.3 with the obvious identifications $x_1 = \widetilde{x}_1$ and $x_7 = \widetilde{x}_7$. As indicated in Figure 5.3 (left) we shall regard the edges x_i as being above a crossing, and the edges $\widetilde{x}_i$ as below the crossing.

Assigning a complex-valued variable to each of the edges $x_1, \cdots, x_7, \widetilde{x}_1, \cdots, \widetilde{x}_7$ with the same label as the edge itself, Hikami and Inoue [HI14, §2.2] consider the

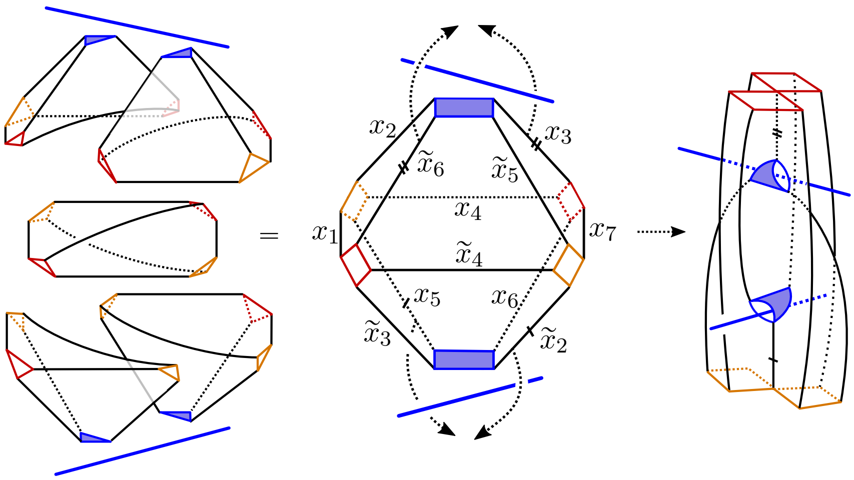


Figure 5.2: A truncated octahedron

equation $(\widetilde{x}_1, \cdots, \widetilde{x}_7) = R^{\pm}(x_1, \cdots, x_7)$ where $R^{\pm}$ is a certain operator defined by rational polynomial equations. As we shall see in Section 5.1.3, these equations are equivalent to Ptolemy relations for a particular obstruction cocycle.

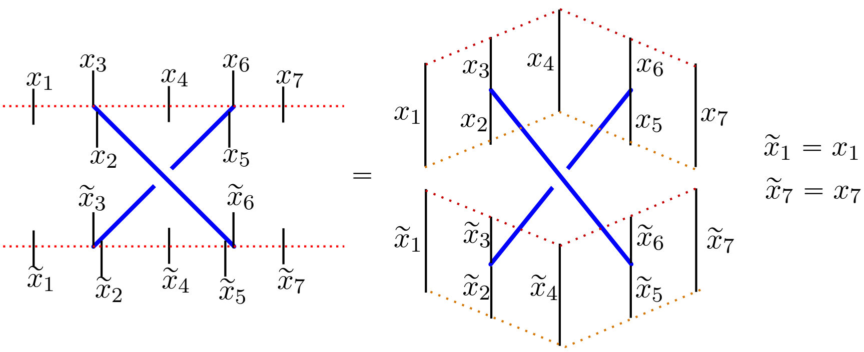


Figure 5.3: Edges of an octahedron at a crossing

Now suppose the knot diagram is given by a braid D with presentation $\sigma_{k_1}^{\epsilon_1} \cdots \sigma_{k_n}^{\epsilon_n}$. (Here σ_{k_i} is the standard generator of the m-braid group and $\epsilon_i \in \{\pm 1\}$.) Similar to the edge-labeling described in the previous paragraph, we label the oriented edges of the octahedral decomposition $\mathcal{O}$ as follows:

1. Draw $n+1$ imaginary horizontal lines on the braid D so that there is only a single crossing between two consecutive lines (see Figures 5.4 and 5.11).

2. As in Figure 5.3 (left), whenever a horizontal line meets the braid D there are two corresponding edges, and whenever a horizontal line meets a region of (the closure of) D, there is one corresponding edge. Since each of the horizontal lines meets the braid m times and the regions $m+1$ times, it corresponds to $3m+1$ edges of $\mathcal{O}$.

3. For the i-th horizontal line we orient the corresponding edges and denote them by $x_1^i, \cdots, x_{3m+1}^i$ as in Figure 5.4, and let $\mathbf{x}^i = (x_1^i, \cdots, x_{3m+1}^i)$.

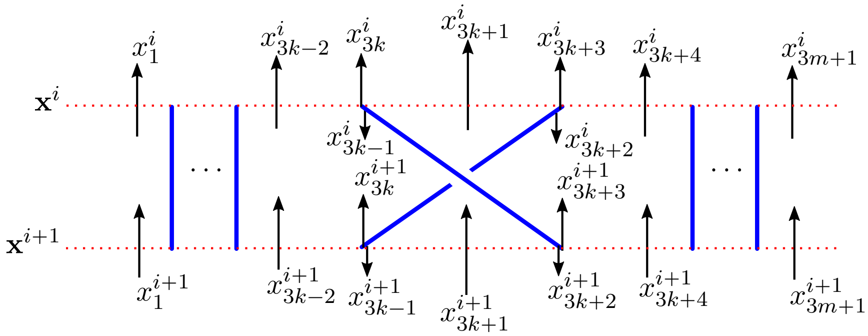


Figure 5.4: Edges of $\mathcal{O}$ around the i-th level of a braid

Note that there are many overlapped labelings; for instance, in Figure 5.4, we have $x_j^i = x_j^{i+1}$ for $j = 1, \cdots, 3k-2$ and $j = 3k+4, \cdots, 3m+1$.

We again assign a complex-valued variable to each oriented edge of $\mathcal{O}$ and denote the variable by the same as the edge itself. Hikami and Inoue [HI14] relate the cluster variables $\mathbf{x}^i = (x_1^i, \cdots, x_{3m+1}^i)$ and $\mathbf{x}^{i+1} = (x_1^{i+1}, \cdots, x_{3m+1}^{i+1})$ by the equation

$$\mathbf{x}^{i+1} = R_{k_i}^{\epsilon_i}(\mathbf{x}^i)$$

for $1 \leqslant i \leqslant n$. Recall that the operator $R_k^{\pm}$ is defined by

$$R_k^{\pm}(x_1, \cdots, x_{3m+1}) = (x_1, \cdots, R^{\pm}(x_{3k-2}, \cdots, x_{3k+4}), x_{3k+5}, \cdots, x_{3m+1}).$$

Note that $R_k^{\pm}$ only affects the variables above and below the k-th crossing.

An initial variable $\mathbf{x}^1$ is called a solution if $\mathbf{x}^1 = \mathbf{x}^{n+1}$. Whenever we have a solution $\mathbf{x}^1 \in \mathbb{C}^{3m+1}$, we shall define the set map

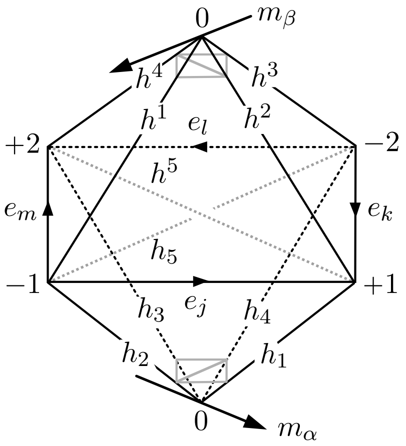
$$c_{\mathbf{x}^1} : \mathcal{O}^1 \to \mathbb{C}$$

by assigning the variable x_j^i to the oriented edge of $\mathcal{O}$ labeled by the same name. The fact that this assignment respects the face identifications in $\mathcal{O}$ follows directly from the definitions of $R_k^{\pm}$ and $R^{\pm}$.

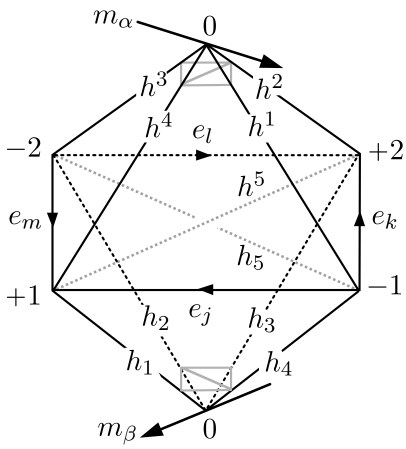
5.1.3 The obstruction cocycle

Let $\mathcal{T}$ be the ideal triangulation of $S^3 \backslash (K \cup \{p, q\})$ obtained by decomposing each octahedron of $\mathcal{O}$ into 5 ideal tetrahedra as in Figure 5.2 (left). As explained earlier this induces a triangulation of the boundary of M. We now define a cocycle $\epsilon \in Z^1(\partial M; \{\pm 1\})$ on ∂M by assigning signs to the short edges of the truncated tetrahedra. Note that each short edge either lies in the top/bottom of a truncated octahedron, or on one of the sides. We shall call the edges *top/bottom-edges* or *side-edges* accordingly. We assign signs to the top/bottom edges as indicated in Figure 5.5 and assign $+1$ do all of the side edges. This is clearly a cocycle, which respects the face pairings and thus gives rise to a cocycle in $\epsilon \in Z^1(\partial M; \{\pm 1\})$ as desired. We stress that ϵ depends on the decomposition of M, in particular the choice of a braid D representing K.

The cocycle ϵ is illustrated in 5.6, where μ and λ_{bf} denote the meridian and black-board framed longitude of the knot K, respectively. In particular, ϵ induces the homomorphism $\bar{\epsilon} : \pi_1(\nu(K)) \to \{\pm 1\}$ that maps μ to -1 and λ_{bf} to 1.



(a) Positive crossing



(a) Negative crossing

Figure 5.5: An ideal octahedron at a crossing

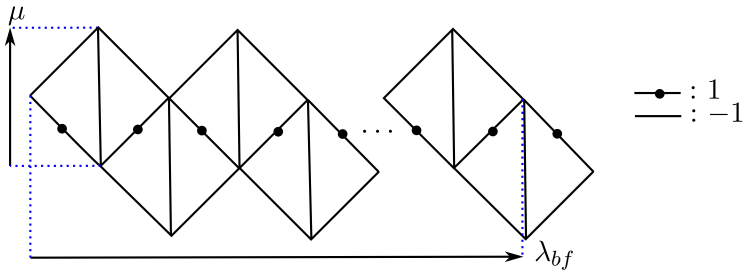


Figure 5.6: Configuration of ϵ on the boundary torus

5.1.4 Proof of Theorem 1.3.2

Let us consider an octahedron of $\mathcal{O}$. We index the vertices by $\{0, \cdots, 5\}$ and denote the oriented edges as in Figure 5.5. Let us compute the ϵ-deformed Ptolemy equation. For example, the tetrahedron with vertices $\{0, 3, 4, 5\}$ in Figure 5.5(a) gives $x_2y_1 = x_3x_4 + x_1x_3$. Similar computations give:

$$\begin{aligned}
\{0,3,4,5\}: \quad x_2y_1 &= x_3x_4 + x_1x_3 \\
\{1,2,3,5\}: \quad x_6y_2 &= x_5x_7 + x_4x_5 \\
\{2,3,4,5\}: \quad x_4\widetilde{x}_4 &= x_1x_7 + y_1y_2 \\
\{0,2,4,5\}: \quad \widetilde{x}_5y_1 &= x_3\widetilde{x}_4 + x_3x_7 \\
\{1,2,3,4\}: \quad \widetilde{x}_3y_2 &= x_5\widetilde{x}_4 + x_1x_5
\end{aligned}$$

for Figure 5.5(a) and

$$\begin{aligned}
\{0,2,4,5\}: \quad y_1x_5 &= x_4x_6 + x_6x_7 \\
\{1,2,3,4\}: \quad x_3y_2 &= x_1x_2 + x_2x_4 \\
\{2,3,4,5\}: \quad x_4\widetilde{x}_4 &= y_1y_2 + x_1x_7 \\
\{0,3,4,5\}: \quad \widetilde{x}_2y_1 &= x_6\widetilde{x}_4 + x_1x_6 \\
\{1,2,3,5\}: \quad \widetilde{x}_6y_2 &= x_2x_7 + x_2\widetilde{x}_4
\end{aligned}$$

for Figure 5.5(b). Considering $x_1, \cdots, x_7$ as given variables, we have

$$(y_1, y_2) = \left(\frac{x_3(x_1+x_4)}{x_2}, \ \frac{x_5(x_4+x_7)}{x_6}\right) \tag{5.1.1}$$

$$(\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5) = \begin{pmatrix} \dfrac{x_1x_3x_5 + x_3x_4x_5 + x_1x_2x_6}{x_2x_4} \\ \dfrac{x_1x_3x_4x_5 + x_3x_4^2x_5 + x_1x_3x_5x_7 + x_3x_4x_5x_7 + x_1x_2x_6x_7}{x_2x_4x_6} \\ \dfrac{x_3x_4x_5 + x_3x_5x_7 + x_2x_6x_7}{x_4x_6} \end{pmatrix}^T$$

for Figure 5.5(a) and

$$(y_1, y_2) = \left(\frac{x_6(x_4+x_7)}{x_5}, \ \frac{x_2(x_1+x_4)}{x_3}\right) \tag{5.1.2}$$

$$(\widetilde{x}_2, \widetilde{x}_4, \widetilde{x}_6) = \begin{pmatrix} \dfrac{x_1x_3x_5 + x_1x_2x_6 + x_2x_4x_6}{x_3x_4} \\ \dfrac{x_1x_2x_4x_6 + x_2x_4^2x_6 + x_1x_3x_5x_7 + x_1x_2x_6x_7 + x_2x_4x_6x_7}{x_3x_4x_5} \\ \dfrac{x_2x_4x_6 + x_3x_5x_7 + x_2x_6x_7}{x_4x_5} \end{pmatrix}^T$$

for Figure 5.5(b).

Letting $\widetilde{x}_1 = x_1,\ \widetilde{x}_2 = x_5,\ \widetilde{x}_6 = x_3,\ \widetilde{x}_7 = x_7$ for Figure 5.5(a) and $\widetilde{x}_1 = x_1,\ \widetilde{x}_3 = x_6,\ \widetilde{x}_5 = x_2,\ \widetilde{x}_7 = x_7$ for Figure 5.5(b), we obtain

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \\ \widetilde{x}_4 \\ \widetilde{x}_5 \\ \widetilde{x}_6 \\ \widetilde{x}_7 \end{pmatrix}^T = \begin{pmatrix} x_1 \\ x_5 \\ \frac{x_1x_3x_5+x_3x_4x_5+x_1x_2x_6}{x_2x_4} \\ \frac{x_1x_3x_4x_5+x_3x_4^2x_5+x_1x_3x_5x_7+x_3x_4x_5x_7+x_1x_2x_6x_7}{x_2x_4x_6} \\ \frac{x_3x_4x_5+x_3x_5x_7+x_2x_6x_7}{x_4x_6} \\ x_3 \\ x_7 \end{pmatrix}^T = R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}^T \qquad (5.1.3)$$

for Figure 5.5(a) and

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \\ \widetilde{x}_4 \\ \widetilde{x}_5 \\ \widetilde{x}_6 \\ \widetilde{x}_7 \end{pmatrix}^T = \begin{pmatrix} x_1 \\ \frac{x_1x_3x_5+x_1x_2x_6+x_2x_4x_6}{x_3x_4} \\ x_6 \\ \frac{x_1x_2x_4x_6+x_2x_4^2x_6+x_1x_3x_5x_7+x_1x_2x_6x_7+x_2x_4x_6x_7}{x_3x_4x_5} \\ x_2 \\ \frac{x_3x_5x_7+x_2x_4x_6+x_2x_6x_7}{x_4x_5} \\ x_7 \end{pmatrix}^T = R^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}^T \tag{5.1.4}$$

for Figure 5.5(b). The equations (5.1.3) and (5.1.4) exactly coincide with the definition of the operators $R^{\pm}$ in [HI15]. See [HI15, Equation (2.17)].

Now let D be a braid of length n and width m. Let $c_{\mathbf{x}^1} : \mathcal{O}^1 \to \mathbb{C}$ be the set map induced from a solution $\mathbf{x}^1 \in \mathbb{C}^{3m+1}$ as in Section 5.1.2. Recall that $\mathcal{T}$ has two additional edges per crossing compared to $\mathcal{O}$. We extend the set map to $c_{\mathbf{x}^1} : \mathcal{T}^1 \to \mathbb{C}$ by defining the values on the added edges using the equations (5.1.1) and (5.1.2). We say that a solution $\mathbf{x}^1$ is *non-degenerate* if

$$c_{\mathbf{x}^1}(e) \neq 0$$

for all $e \in \mathcal{T}^1$. One can easily check from the equations (5.1.1) and (5.1.2) that this is equivalent to the following.

Definition 5.1.1. A solution $\mathbf{x}^1$ is said to be *non-degenerate* if every cluster variable $\mathbf{x}^i = (x^i_1, \cdots, x^i_{3m+1})$ satisfies $x^i_j \neq 0$ for all $1 \leqslant j \leqslant 3m+1$ and $x^i_{3j-2} \neq -x^i_{3j+1}$ for all $1 \leqslant j \leqslant m$.

The previous computation in this section tells us that the set map $c_{\mathbf{x}^1} : \mathcal{T}^1 \to \mathbb{C}\backslash\{0\}$ induced from a non-degenerated solution $\mathbf{x}^1$ is a point of the ϵ-deformed

Ptolemy variety $P_\epsilon(\mathcal{T})$. We have thus proven (recall Proposition 2.2.1):

Proposition 5.1.1. A non-degenerate solution $\mathbf{x}^1$ induces a $(\mathrm{PSL}(2,\mathbb{C}),P)$-representation $\rho_{\mathbf{x}^1} : \pi_1(S^3\backslash K) = \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ (up to conjugation) whose obstruction class is $\overline{\epsilon}_D(\lambda) \in \{\pm 1\} \simeq H^2(M,\partial M;\{\pm 1\})$, where λ is the canonical longitude.

Proposition 5.1.2. Let D be a braid of length n representing a knot. Then $\overline{\epsilon}_D(\lambda)$ is $(-1)^n$ under the isomorphism $H^2(M,\partial M;\{\pm 1\}) \simeq \{\pm 1\}$.

Proof. Recall Section 5.1.3 that we have $\overline{\epsilon}_D(\mu) = -1$ and $\overline{\epsilon}_D(\lambda_{bf}) = 1$ for the meridian μ and blackboard framed longitude λ_{bf}. We thus obtain

$$\begin{aligned}\overline{\epsilon}_D(\lambda) &= \overline{\epsilon}_D(\lambda_{bf})\ \overline{\epsilon}_D(\mu)^{-w(D)} \\ &= \overline{\epsilon}_D(\lambda_{bf})\ \overline{\epsilon}_D(\mu)^{-n} = (-1)^n.\end{aligned}$$

Here $w(D)$ denotes the writhe of the closure of D which is congruent to the length n in modulo 2. $\square$

5.2 The existence of a non-degenerate solution

Let $\widetilde{M}$ be the universal cover of $M = S^3\backslash\nu(K \cup \{p,q\})$ and $\widehat{\widetilde{M}}$ be the space obtained from $\widetilde{M}$ by collapsing each boundary component to a point. We denote by $I(\widetilde{M})$ the set of these points. Note that $\pi_1(M)$ acts on $I(\widetilde{M})$.

Definition 5.2.1. For a $(\mathrm{PSL}(2,\mathbb{C}),P)$-representation $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, a *decoration* $\mathcal{D} : I(\widetilde{M}) \to \mathrm{PSL}(2,\mathbb{C})/P$ is a ρ-equivalent assignment, i.e., $\mathcal{D}(\gamma \cdot v) = \rho(\gamma)\mathcal{D}(v)$ for all $\gamma \in \pi_1(M)$ and $v \in I(\widetilde{M})$.

Recall that $\mathrm{PSL}(2,\mathbb{C})/P$ denotes the (left) P-coset space where P is the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ consisting of upper triangular matrices with ones on the diagonal. We may identify a P-coset gP with a vector $g\binom{1}{0}$ which is well-defined up to sign. In particular, by $\det(gP,hP)$ we mean $\det\left(g\binom{1}{0}, h\binom{1}{0}\right) \in \mathbb{C}/\{\pm 1\}$.

We now fix a braid presentation D of a knot K and let $\mathcal{T}$ be the ideal triangulation of $S^3\backslash(K \cup \{p,q\})$ given as in Section 5.1. For any decoration $\mathcal{D}$ we define an assignment $c : \mathcal{T}^1 \to \mathbb{C}/\{\pm 1\}$ by

$$c(e) = \det\left(\mathcal{D}(v_1), \mathcal{D}(v_2)\right)$$

for $e \in \mathcal{T}^1$ where v_1 and $v_2 \in I(\widetilde{M})$ are endpoints of a lift of e. Note that $c(e)$ does not depend on the choice of a lift of e, since $\mathcal{D}$ is ρ-equivariant.

Proposition 5.2.1. For a non-trivial $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, there exists a decoration $\mathcal{D}$ such that the induced assignment c satisfies $c(e) \neq 0$ for all $e \in \mathcal{T}^1$.

The proof of Proposition 5.2.1 relies on the following basic facts: (i) every edge of $\mathcal{T}$ are connected to either p or q; (ii) a decoration on the lifts of p and q can be chosen freely and independently (respecting ρ-equivalence only). The observation that (i) and (ii) implies Proposition 5.2.1 was first pointed out to the authors by Seonhwa Kim. We also note that there are edges connecting p (or q) to itself and this is the reason why we can not detect the trivial representation. Namely, these edges become generators in the Wirtinger representation (see Figure 4.9 (left)) and thus the image of the generators under ρ must be non-trivial.

Remark 5.2.1. In order for fact (i) above to hold, it is essential that each octahedron is subdivided into five tetrahedra instead of four. If we use the four tetrahedra per crossing (as in [HI15]) Proposition 5.2.1 may not hold; in particular, it does not hold whenever the closure of D has a kink.

Proposition 5.2.1 implies the existence of a non-degenerate solution desired as in Theorem 1.3.3. More precisely, the following holds.

Theorem 5.2.1. Let $\sigma_D \in Z^2(M, \partial M; \{\pm 1\})$ be the cocycle given as in Section 5.1. If a non-trivial $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation ρ has the obstruction class

$[\sigma_D] \in H^2(M, \partial M; \{\pm 1\})$, then there exists a point $c \in P^{\sigma_D}(\mathcal{T})$ such that ρ_c coincides with ρ up to conjugation.

Proof. Let $\mathcal{D}$ be a decoration as in Proposition 5.2.1. Whenever one chooses a sign of each $c(e)$, it is known that $c : \mathcal{T}^1 \to \mathbb{C}\backslash\{0\}$ is a point of $P^\sigma(\mathcal{T})$ for some $\sigma \in Z^2(M, \partial M; \{\pm 1\})$ such that $\rho_c = \rho$ up to conjugation. In particular, the obstruction class of ρ is $[\sigma] \in H^2(M, \partial M; \{\pm 1\})$. Then the theorem follows from the fact that if σ_0 and $\sigma_1 \in Z^2(M, \partial M; \{\pm 1\})$ satisfy $[\sigma_0] = [\sigma_1]$, then two varieties $P^{\sigma_0}(\mathcal{T})$ and $P^{\sigma_1}(\mathcal{T})$ are canonically isomorphic. □

As we computed in Section 5.1.4, the class $[\sigma_D]$ viewed as an element of $\{\pm 1\}$ coincides with $(-1)^n$ where n is the length of D. We therefore obtain Theorem 1.3.3 as a consequence of Theorem 5.2.1.

5.2.1 Proof of Proposition 5.2.1

We first consider edges, say $e_1, \cdots, e_m$, of $\mathcal{T}$ that join p and q. We orient these edges from q to p. We choose a lift $\widetilde{e}_j$ of each e_j so that their terminal points agree as in Figure 4.8. We denote by $\widetilde{p}$ the terminal point and by $\widetilde{q}_j$ the initial point of $\widetilde{e}_j$. From ρ-equivariance of $\mathcal{D}$, we have

$$\mathcal{D}(\widetilde{q}_j) = \rho(g)\mathcal{D}(\widetilde{q}_k)$$

for some $g \in \pi_1(M)$. From elementary covering theory one can check that if $e_j \cup e_k$ wraps an arc of K as in Figure 5.7, then the loop g should be the Wirtinger generator corresponding to the arc. Note that $c(e_k) \neq 0$ if and only if $\det(\mathcal{D}(\widetilde{p}), \mathcal{D}(\widetilde{q}_k)) \neq 0$.

We then consider edges of $\mathcal{T}$ that are connected to the knot K; for example, edges x and y as in Figure 5.7. We consider an ideal triangle in $S^3\backslash(K \cup \{p, q\})$ with edges x, y, e_k as in Figure 5.7, and its lift so that p corresponds to the point $\widetilde{p}$. We denote the edges of the lift by $\widetilde{x}, \widetilde{y}, \widetilde{e}_k$. Since the terminal point, $\widetilde{r}$, of $\widetilde{x}$

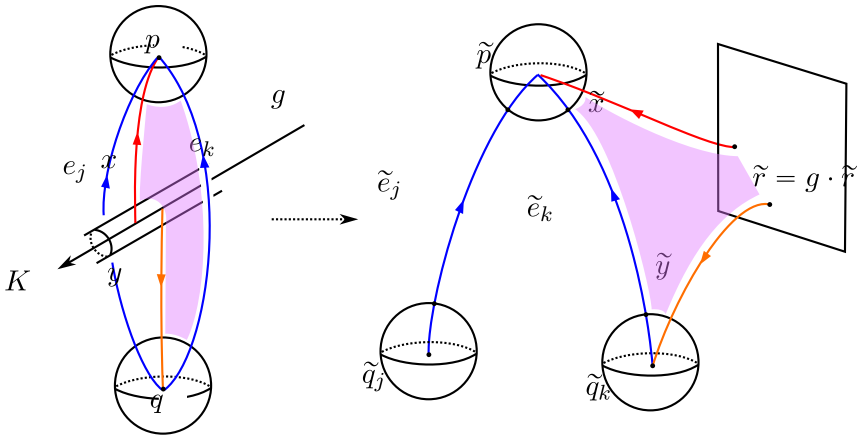


Figure 5.7: Local configuration of a lift.

(or $\widetilde{y}$) is fixed by the Wirtinger generator g, we obtain

$$\mathscr{D}(\widetilde{r}) = \mathscr{D}(g \cdot \widetilde{r}) = \rho(g)\mathscr{D}(\widetilde{r}).$$

Since $\mathrm{tr}(\rho(g)) = \pm 2$ and $\rho(g) \neq \mathrm{Id}$, otherwise ρ should be a trivial representation, $\rho(g)$ has a unique eigenvector up to scaling. It thus follows that $c(x) = \det(\mathscr{D}(\widetilde{p}), \mathscr{D}(\widetilde{r})) \neq 0$ if and only if $\mathscr{D}(\widetilde{p})$ is not an eigenvector of $\rho(g)$. Similarly, $c(y) \neq 0$ if and only if $\mathscr{D}(\widetilde{q}_k)$ is not an eigenvector of $\rho(g)$.

We finally consider edges of $\mathscr{T}$ joining q (or p) to itself; for example, an edge x as in Figure 5.8. We consider an ideal triangle in $S^3 \backslash (K \cup \{p, q\})$ with edges e_j, e_k, x as in Figure 5.8, and its lift so that p corresponds to the point $\widetilde{p}$. We denote the edges of the lift by $\widetilde{e}_j, \widetilde{e}_k, \widetilde{x}$. It directly follows that $c(x) \neq 0$ if and only if

$$\det(\mathscr{D}(\widetilde{q}_j), \mathscr{D}(\widetilde{q}_k)) = \det(\rho(g)\mathscr{D}(\widetilde{q}_k), \mathscr{D}(\widetilde{q}_k)) \neq 0.$$

Again, this is equivalent to the condition that $\mathscr{D}(\widetilde{q}_k)$ is not an eigenvector of $\rho(g)$.

Let us sum up the conditions. To be precise, we enumerate the Wirtinger

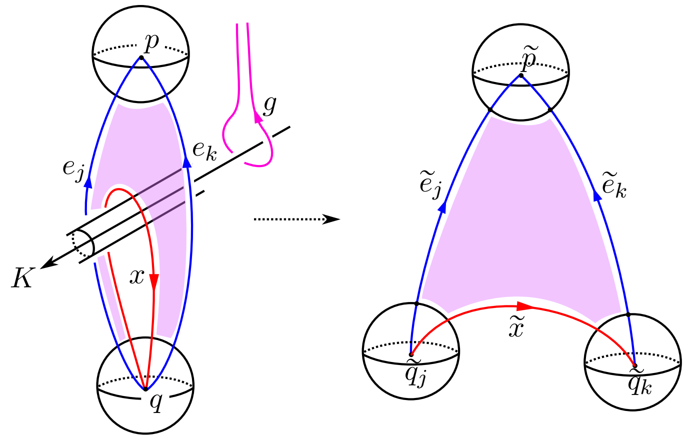


Figure 5.8: Local configuration of a lift.

generators by $g_1, \cdots, g_l$. Our desired decoration as in Proposition 5.2.1 should satisfy (i) $\det(\mathcal{D}(\widetilde{p}), \mathcal{D}(\widetilde{q}_j)) \neq 0$; (ii) $\mathcal{D}(\widetilde{p})$ is not an eigenvector of $\rho(g_i)$; (iii) $\mathcal{D}(\widetilde{q}_j)$ is not an eigenvector of $\rho(g_i)$ for all $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant l$. Since we can choose $\mathcal{D}(\widetilde{p})$ and one of $\mathcal{D}(\widetilde{q}_j)$'s freely, such a decoration exists.

5.2.2 Explicit computation from a representation

Let D be a braid presentation of a knot K and let $\rho : \pi_1(S^3 \backslash K) \to \mathrm{PSL}(2, \mathbb{C})$ be a non-trivial $(\mathrm{PSL}(2, \mathbb{C}), P)$-representation whose obstruction class is $(-1)^n$, where n is the length of D. We devote this subsection to present an explicit formula for computing a solution.

Let $\widetilde{\rho}$ be an -lift of ρ satisfying

$$\widetilde{\rho}(\mu) = \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix} \neq -\mathrm{Id} \quad \text{and} \quad \widetilde{\rho}(\lambda) = \begin{pmatrix} (-1)^n & * \\ 0 & (-1)^n \end{pmatrix}$$

(recall Proposition 2.2.1). We index the regions of the closure of D by $1 \leqslant j \leqslant n+2$ and the arcs by $1 \leqslant i \leqslant n$. We then assign a non-zero column vector V_j to the j-th region so that these vectors satisfy

$$V_{j_2} = \widetilde{\rho}(g_i)^{-1} V_{j_1} \tag{5.2.5}$$

for Figure 4.11 (left) where m_i is the Wirtinger generator corresponding to the i-th arc. The region-colorings are well-determined whenever an initial vector is chosen arbitrarily. Remark that V_j corresponds to $\mathcal{D}(\widetilde{q}_j)$ in Section 5.2.1.

We also assign a non-zero column vector H_i to the i-th arc so that these vectors satisfy $\widetilde{\rho}(g_i)H_i = -H_i$ for $1 \leqslant i \leqslant m$ (recall that the eigenvalue of $\widetilde{\rho}(g_i)$ is -1) and

$$H_{i_3} = \widetilde{\rho}(g_{i_2})^{-1} H_{i_1} \tag{5.2.6}$$

for Figure 5.9 (right). We remark that the fact that the eigenvalue of $\widetilde{\rho}(\lambda_{bf})$ is 1 (equivalently, the eigenvalue of $\widetilde{\rho}(\lambda)$ is $(-1)^n$) is required here.

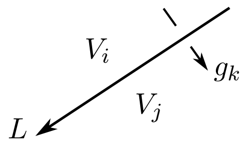


Figure 5.9: Rules for region- and arc-colorings.

Recall that the octahedral decomposition $\mathcal{O}$ has $3n+2$ edges; (i) n of them, called *over-edges*, stand above the knot; (ii) other n of them, called *under-edges*, stand below the knot; (iii) last $n+2$ of them, called *regional edges*, stand on the regions. See Figure 5.10. We choose an additional non-zero column vector W (which corresponds to $\mathcal{D}(\widetilde{p})$ in Section 5.2.1) and define the set map $c : \mathcal{O}^1 \to \mathbb{C}$ as follows.

(i) $c(e) = \det(H_i, W)$ if e is the over-edge standing over the i-th arc;

(ii) $c(e) = \det(V_j, H_i)$ if e is the under-edge standing below the i-th arc whose left-side region is indexed by j;

(iii) $c(e) = \det(V_j, W)$ if e is the regional edge corresponding the j-th region.

Here we oriented the edge e as in Figure 5.10.

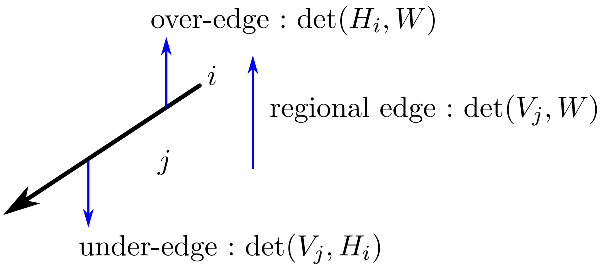


Figure 5.10: Edges of $\mathcal{O}$ with c-values.

We again extend the above set map to $c : \mathcal{T}^1 \to \mathbb{C}$ by using the equations (5.1.1) and (5.1.2). As we showed in Section 5.2.1 for a generic choice of W and V_j's, we have $c(e) \neq 0$ for all $e \in \mathcal{T}^1$.

Example 5.2.1 (The 4_1 knot with a kink)**.** Let us consider a braid of the knot 4_1 as in Figure 5.11. The geometric representation ρ lifts to an -representation

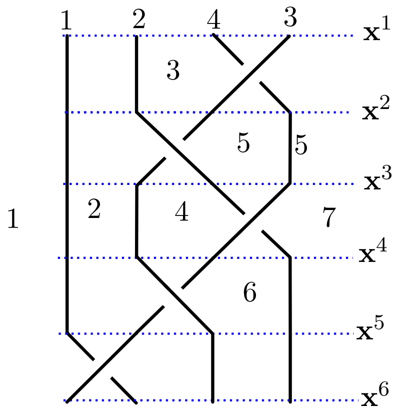


Figure 5.11: A braid presentation of the 4_1 knot.

$\widetilde{\rho}$ such that

$$\widetilde{\rho}(g_1) = \widetilde{\rho}(g_2) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}(g_3) = \begin{pmatrix} -1 & 0 \\ -\lambda & -1 \end{pmatrix}$$
$$\widetilde{\rho}(g_4) = \begin{pmatrix} -1-\lambda & \lambda \\ -\lambda & -1+\lambda \end{pmatrix}, \quad \widetilde{\rho}(m_4) = \begin{pmatrix} -2 & \lambda \\ -1+\lambda & 0 \end{pmatrix}$$

where $\lambda^2 - \lambda + 1 = 0$.

We enumerate the arcs and regions of the closure of the braid as in Figure 5.11. Choosing the vector $H_1 = \binom{1}{0}$, the equation (5.2.6) gives

$$H_2 = \widetilde{\rho}(g_2)^{-1} H_1 = \tbinom{-1}{0}, \quad H_3 = \widetilde{\rho}(g_5)^{-1} H_2 = \tbinom{0}{-1+\lambda}$$
$$H_4 = \widetilde{\rho}(g_2) H_3 = \tbinom{1-\lambda}{1-\lambda}, \quad H_5 = \widetilde{\rho}(g_3)^{-1} H_4 = \tbinom{-1+\lambda}{\lambda}.$$

Similarly, letting the vector $V_1 = \binom{\alpha}{\beta}$ for some $\alpha, \beta \in \mathbb{C}$, the equation (5.2.5) gives

$$V_2 = \widetilde{\rho}(g_1)^{-1} V_1 = \tbinom{-\alpha+\beta}{-\beta}, \qquad V_3 = \widetilde{\rho}(g_2)^{-1} V_2 = \tbinom{\alpha-2\beta}{\beta}$$
$$V_4 = \widetilde{\rho}(g_4)^{-1} V_2 = \tbinom{\alpha(1-\lambda)+\beta(-1+2\lambda)}{-\alpha\lambda+\beta(1+2\lambda)}, \quad V_5 = \widetilde{\rho}(g_3)^{-1} V_3 = \tbinom{-\alpha+2\beta}{\alpha\lambda-\beta(1+2\lambda)}$$
$$V_6 = \widetilde{\rho}(g_5)^{-1} V_4 = \tbinom{\alpha(-1+\lambda)+\beta(2-3\lambda)}{\alpha\lambda-\beta(1+3\lambda)}, \quad V_7 = \widetilde{\rho}(g_5)^{-1} V_5 = \tbinom{\alpha(1-\lambda)+\beta(-2+3\lambda)}{-\alpha(1+\lambda)+2\beta(2+\lambda)}.$$

Then finally, letting the vector $W = \binom{\gamma}{1}$ for some $\gamma \in \mathbb{C}$, we obtain the cluster variables $\mathbf{x}^1, \cdots, \mathbf{x}^5$ as follows. We note that a generic choice for α, β, and γ

gives a non-degenerate solution. Here we abbreviate $\det(\cdot,\cdot)$ by $|\cdot,\cdot|$.

$$\mathbf{x}^1 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_3,H_2| \\ |H_2,W| \\ |V_3,W| \\ |V_6,H_4| \\ |H_4,W| \\ |V_6,W| \\ |V_7,H_3| \\ |H_3,W| \\ |V_7,W| \end{pmatrix}^T = \begin{pmatrix} \alpha-\beta\gamma \\ \beta \\ 1 \\ -\alpha+\beta\gamma+\beta \\ \beta \\ -1 \\ \alpha-\beta(\gamma+2) \\ (\lambda-1)(\alpha-3\beta) \\ (\gamma-1)(\lambda-1) \\ \alpha(-\gamma\lambda+\lambda-1)+\beta(3(\gamma-1)\lambda+\gamma+2) \\ \alpha\lambda-\beta(2\lambda+1) \\ \gamma-\gamma\lambda \\ \alpha((\gamma-1)\lambda+\gamma+1)-\beta(2\gamma(\lambda+2)-3\lambda+2) \end{pmatrix}^T$$

$$\mathbf{x}^2 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_3,H_2| \\ |H_2,W| \\ |V_3,W| \\ |V_5,H_3| \\ |H_3,W| \\ |V_5,W| \\ |V_7,H_5| \\ |H_5,W| \\ |V_7,W| \end{pmatrix}^T = \begin{pmatrix} \alpha-\beta\gamma \\ \beta \\ 1 \\ -\alpha+\beta\gamma+\beta \\ \beta \\ -1 \\ \alpha-\beta(\gamma+2) \\ \lambda^2(-(\alpha-2\beta)) \\ \gamma-\gamma\lambda \\ \beta(2\gamma\lambda+\gamma+2)-\alpha(\gamma\lambda+1) \\ (\lambda-1)(\alpha-3\beta) \\ -\gamma\lambda+\lambda-1 \\ \alpha((\gamma-1)\lambda+\gamma+1)-\beta(2\gamma(\lambda+2)-3\lambda+2) \end{pmatrix}^T$$

$$\mathbf{x}^3 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_4,H_4| \\ |H_4,W| \\ |V_4,W| \\ |V_5,H_2| \\ |H_2,W| \\ |V_5,W| \\ |V_7,H_5| \\ |H_5,W| \\ |V_7,W| \end{pmatrix}^T = \begin{pmatrix} \alpha-\beta\gamma \\ \beta \\ 1 \\ -\alpha+\beta\gamma+\beta \\ (\lambda-1)(-(\alpha-2\beta)) \\ (\gamma-1)(\lambda-1) \\ (\gamma-1)\lambda(\alpha-2\beta)+\alpha-\beta(\gamma+1) \\ \alpha\lambda-\beta(2\lambda+1) \\ -1 \\ \beta(2\gamma\lambda+\gamma+2)-\alpha(\gamma\lambda+1) \\ (\lambda-1)(\alpha-3\beta) \\ -\gamma\lambda+\lambda-1 \\ \alpha((\gamma-1)\lambda+\gamma+1)-\beta(2\gamma(\lambda+2)-3\lambda+2) \end{pmatrix}^T$$

$$\mathbf{x}^4 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_4,H_4| \\ |H_4,W| \\ |V_4,W| \\ |V_6,H_5| \\ |H_5,W| \\ |V_6,W| \\ |V_7,H_3| \\ |H_3,W| \\ |V_7,W| \end{pmatrix}^T = \begin{pmatrix} \alpha-\beta\gamma \\ \beta \\ 1 \\ -\alpha+\beta\gamma+\beta \\ (\lambda-1)(-(\alpha-2\beta)) \\ (\gamma-1)(\lambda-1) \\ (\gamma-1)\lambda(\alpha-2\beta)+\alpha-\beta(\gamma+1) \\ -\beta \\ -\gamma\lambda+\lambda-1 \\ \alpha(-\gamma\lambda+\lambda-1)+\beta(3(\gamma-1)\lambda+\gamma+2) \\ \alpha\lambda-\beta(2\lambda+1) \\ \gamma-\gamma\lambda \\ \alpha((\gamma-1)\lambda+\gamma+1)-\beta(2\gamma(\lambda+2)-3\lambda+2) \end{pmatrix}^T$$

$$\mathbf{x}^5 = \begin{pmatrix} |V_1, W| \\ |V_2, H_1| \\ |H_1, W| \\ |V_2, W| \\ |V_3, H_1| \\ |H_1, W| \\ |V_3, W| \\ |V_6, H_4| \\ |H_4, W| \\ |V_6, W| \\ |V_7, H_3| \\ |H_3, W| \\ |V_7, W| \end{pmatrix}^T = \begin{pmatrix} \alpha - \beta\gamma \\ \beta \\ 1 \\ -\alpha + \beta\gamma + \beta \\ -\beta \\ 1 \\ \alpha - \beta(\gamma + 2) \\ (\lambda - 1)(\alpha - 3\beta) \\ (\gamma - 1)(\lambda - 1) \\ \alpha(-\gamma\lambda + \lambda - 1) + \beta(3(\gamma - 1)\lambda + \gamma + 2) \\ \alpha\lambda - \beta(2\lambda + 1) \\ \gamma - \gamma\lambda \\ \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2) \end{pmatrix}^T$$

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